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APPLICATION DE L'EFFET MAGNÉTO-OPTIQUE DE FARADAY À L'ÉTUDE DES PROPRIÉTÉS MAGNÉTIQUES DES COUCHES MINCES DES FERROMAGNÉTIQUES

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(Manuscrit reçu le 5 mars 1959)

L'auteur a employé la méthode magnéto-optique de l'effet Faraday pour étudier les propriétés magnétiques des lames minces des métaux ferromagnétiques, obtenues par évaporation dans un vide poussé. On a constaté que la valeur de l'aimantation des lames minces de fer, de nickel et de cobalt diminue, à partir d'une épaisseur critique, par rapport à celle des échantillons massifs. Les résultats obtenus confirment les théories de Klein-Smith et de Glass-Klein sur l'aimantation spontanée des lames minces ferromagnétiques.

I. Introduction

Depuis quelques années l'intérêt suscité par les propriétés des lames minces s'est accru considérablement, tant du point de vue expérimental que théorique. Toutefois, les résultats antérieurs diffèrent fortement par suite de la technique encore défectueuse de confection des lames minces. Ce n'est qu'à partir de la technique moderne du vide que l'on constate un progrès considérable dans ce domaine. Néanmoins, les divergences des propriétés magnétiques des lames minces vis-à-vis de celles des ferromagnétiques massifs continuent d'être l'objet des recherches.

Le but du travail présent consiste à donner une explication des divergences entre les résultats des différents auteurs (Kundt 1884, Du Bois 1887, Behrens 1907, Cau 1929, König 1948) qui utilisaient la méthode magnéto-optique de l'effet Faraday dans leurs études des propriétés magnétiques des métaux ferromagnétiques, et à comparer les résultats expérimentaux avec les théories de Klein-Smith (1951), Glass-Klein (1958) et Valenta (1957) concernant l'aimantation de saturation dans les lames minces ferromagnétiques.

Les méthodes de mesure couramment employées dans le domaine des lames minces utilisent l'effet Faraday à cause de la valeur très considérable de la rotation du plan de polarisation dans les ferromagnétiques. P. ex. dans le fer, la rotation calculée par 1 cm d'épaisseur atteint une valeur de 387000°. Ainsi, cette méthode doit présenter un intérêt tout particulier en ce qui concerne l'étude des propriétés magnétiques dans les lames les plus minces des matériels ferromagnétiques.

II. Résultats antérieurs

Depuis l'époque de Kundt auquel est due en 1884 la découverte de la rotation du plan de polarisation de la lumière dans des lames minces des métaux ferromagnétiques soumis au champ magnétique (effet Faraday), le nombre des travaux parus dans ce domaine n'est pas considérable (Du Bois 1887, Behrens 1907, Cau 1929, König 1948). Les travaux de Kundt furent continués par du Bois, qui trouva que l'angle de rotation du plan de polarisation de la lumière passant par une lame mince de ferromagnétique est proportionnel à l'aimantation M et à l'épaisseur d de la lame

$$\alpha = C_k Md \tag{1}$$

On désigne le coefficient C_k constante de Kundt.

Afin d'obtenir des lames minces des ferromagnétiques, on utilisait l'une des trois méthodes suivantes:

électrolyse
pulvérisation cathodique
sublimation dans le vide élevé.

Les deux premières ne préservent pas d'une oxydation rapide le dépôt en train de formation, surtout si son épaisseur n'est que de l'ordre de quelques dizaines de Å. C'est à cause de cela que les résultats des auteurs ayant utilisé des couches électrolytiques ou pulvérisées présentent des erreurs importantes. Ainsi, nous ne considérons que les résultats des auteurs qui ont procédé par sublimation dans le vide élevé. Nonobstant le fait que tous ces auteurs aient étudié des lames sublimées dans le vide, les résultats de leurs mesures magnéto-optiques pour des lames minces de fer varient fortement. Cau (1929) obtient une rotation maximum (saturation) dans un champ de 10 KOe, tandis que König (1948) obtient la saturation dans des lames minces de fer avec un champ de 21 KOe. König (1948), de ses recherches très complètes et témoignant de conditions expérimentales idéales (toutes ses recherches furent faites dans le vide) conclue que la vraie cause des discordances entre les résultats antérieurs réside dans l'oxydation de la couche. En étudiant simultanément la structure des couches par la méthode de diffraction d'électrons, il a pu constater que les dépôts de fer obtenus par sublimation dans un vide même excellent, mais dans un procédé lent (temps de vaporisation de l'ordre de minutes), et condensés sur support non chauffé, présentent une structure granulée extrêmement fine, avec de propriétés qui évoluent dans le temps, et une facilité très grande d'oxydation. Pour ces couches König trouva que la rotation diminuait d'environ 15% au cours d'une heure; en utilisant la diffraction d'électrons, il montra que cette diminution résulte de l'oxydation des couches. Par contre, des couches obtenues par évaporation rapide (vitesse d'accroissement d'épaisseur dépassant 50 Å/sec) et condensées sur support chauffé sont stables; leurs propriétés n'évoluent pas si vite et elles ne subissent qu'une oxydation très faible au bout de quelques mois. Cette constatation est d'une grande portée pour les recher-

ches par effet Faraday. Toutefois, la méthode de König qui exige que le dispositif utilisé ainsi que l'électroaimant soient placés dans le vide, soulève de difficultés du point de vue expérimental.

König (1948) étudia aussi la rotation maximum (en saturation magnétique) du plan de polarisation dans le fer en fonction de l'épaisseur de la lame. Fig. 1 montre les résultats de ses mesures. Il trouve que la fonction est linéaire dans tout l'intervalle

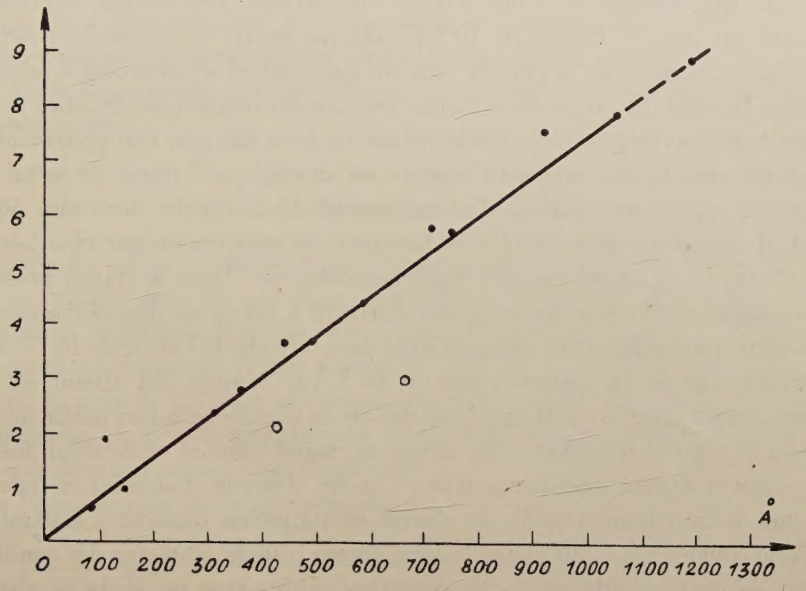

Fig. 1. La valeur doublée de l'angle de rotation du plan de polarisation en fonction de l'épaisseur de couche, pour le fer, dans un champ de 24 KOe (König[5])

des épaisseurs; donc, il ne constate pas de chute de l'aimantation pour les couches les plus minces prédite par les théories plus récentes de Klein-Smith (1951) et d'autres (Glass-Klein 1958, Valenta 1957).

Behrens (1907) et Cau (1929) employaient la sublimation lente et leurs mesures magnéto-optiques étaient effectuées en présence de l'air; ainsi, leurs résultats, qui diffèrent entre eux, ne sont pas reproductibles. König (1948), bien qu'il eût fait toutes ses mesures dans le vide, ne constata pas de diminution de l'aimantation pour les lames les plus minces; il est vraisemblable que ceci soit dû au fait qu'il déterminait leur épaisseur par la méthode gravimétrique en admettant que de telles lames ont la même densité que la matériel massif. Les travaux expérimentaux de Rouard (1956), Schopper (1952) et Zavieta (1956) ont démontré que leur densité diffère de celle du matériel massif à partir d'une épaisseur de quelques milliers de Å. Soit d_1 l'épaisseur obtenue par pesage, et d_2 celle que donne la méthode interférométrique; alors, le quotient d_1/d_2 est dit coefficient de remplissage. Sa valeur est de 0,79 pour une épaisseur 172 Å et de 0,91 pour 856 Å.

III. Partie expérimentale

Comme nous l'avons dit, dans l'étude rigoureuse de l'effet Faraday magnéto-optique dans les ferromagnétiques (dans l'intervalle des plus petites épaisseurs de couche), c'est la méthode de confection de la couche qui joue un rôle de tout premier ordre. La déposition de la couche par sublimation doit être effectuée dans un vide aussi parfait que possible et à une vitesse considérable. Les calculs montrent que, même dans un vide de l'ordre de 10^{-6} Tr, chaque atome de la couche fraîchement déposée subit des chocs de la part de ceux du gaz résiduel s'y trouvant à raison d'un choc toutes les quelques secondes. Comme les surfaces métalliques fraîches se distinguent par leur pouvoir très élevé d'adsorption vis-à-vis des gaz, l'on pourra admettre que chaque atome du gaz frappant la surface est absorbé par l'atome du métal. Donc, en évaporant rapidement (vitesse d'accroissement de la couche dépassant 50 Å par seconde), il devient possible de réduire fortement la quantité de gaz adsorbée par la couche mince, ce qui a été constaté expérimentalement. Dans le travail présent, un soin tout spécial a été mis à construire un dispositif à vide pour déposition rapide des couches minces des métaux ferromagnétiques dans un vide de l'ordre de 10^{-5}—10^{-6} Tr. Un accroissement de la couche dépassant 60 Å par seconde fut atteint en faisant passer un courant fort (75 à 200 amp.) par des fils de chauffe en ruban mince de molybdène. Afin d'éviter la formation d'un alliage du métal évaporé et de celui des fils de chauffe, ceux-ci étaient recouverts d'une couche d'oxyde d'aluminium (alundum) ou d'oxyde de béryllium. Des fils de chauffe en tungstène recouverts électrolytiquement d'une couche mince du métal évaporé furent utilisés, ainsi que des conducteurs consistant de quelques fils minces de tungstène tordus avec un fil de ce métal. Ces façons d'agir permettaient de déposer des couches rapidement. Le support sur lequel se formait la couche était tenu à 150—300°C afin d'effectuer son dégazage et dans le but d'améliorer l'homogénéité de la structure de la couche. Ce support fut formé par des couvre-objets de microscope soigneusement lavés et séchés au vide.

L'analyse spectrale quantitative des métaux utilisés ne décela que des additions peu importantes d'autres éléments chimiques, notamment dans le fer: Ni 0,077%, C 0,15%, Si 0,28%, Cr 0,3%, Mn 0,8% et, dans le nickel: Fe 0,21%, Si 0,10%, Cu 0,08%, Mg 0,82%.

Pour déterminer l'épaisseur des lames minces on employa la méthode microscopique d'interférence, dont le degré de précision pour des épaisseurs de 50 Å à 500 Å est de ± 15 Å. Afin de mesurer l'épaisseur, deux lames furent préparées simultanément sur deux couvre-objets dans des conditions identiques. L'une servait à la mesure de l'épaisseur, et l'autre à déterminer la rotation. Comme essai de contrôle, les deux mesures furent effectuées sur le même couvre-objet.

[1] Cette méthode, avec une discussion détaillée de son dégré de précision, a été décrite par Z. Godziński et W. Szczerbski dans „Zeszyty Naukowe Politechniki Wrocławskiej" (Cahiers scientifiques de l'Ecole Polytechnique de Wrocław") 5, 47, 1955.

Le dispositif de mesure pour l'effet magnéto-optique Faraday se composait d'un polarimètre (Hilger) avec échelle projetée donnant une précision de lecture de 0,005°, et d'un électroaimant (Justi 1945) spécialement construit donnant de champs de l'ordre de 25 KOe avec des courants relativement faibles (8 amp.). L'enroulement de l'électroaimant était placé sur des châssis refroidis à l'eau, de manière que même au bout de quelques heures de travail la température dans la fente ne subissait pas de changement. Ceci est d'une importance toute spéciale dans les mesures magnétiques des lames les plus minces pour lesquelles il y a un abaissement considérable du point de Curie, constaté expérimental par Drigo (1951) et prévu théoriquement par Heber (1953). Dans les mesures polarimétriques, on utilisait une lampe à sodium comme source lumineuse.

IV. Résultats des théories récentes

D'après les anciennes théories du ferromagnétisme de Weiss (1948) et de Bloch (1932), un cristal à deux dimensions ou une chaîne d'atomes ne devraient pas présenter d'aimantation spontanée. Les couches nonmagnétiques de nickel obtenues expérimentalement par Ingersoll (1925) ainsi que celles de fer de Felici (1944) semblaient en fournir la preuve. Ce n'est que depuis les travaux de König (1948), qui démontra la présence des propriétés magnétiques même dans les lames les plus minces de fer (16 Å), que date une relance des calculs théoriques de l'aimantation spontanée en fonction de l'épaisseur et de la température.

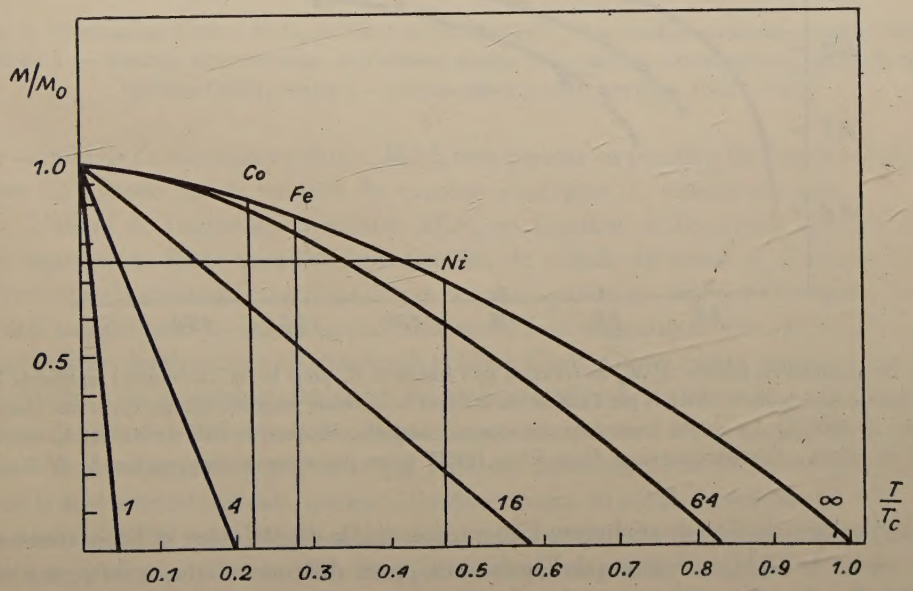

Fig. 2. L'aimantation relative M/M_0 pour différentes valeurs du paramètre G_z nombre des couches atomiques du réseau (d'après Klein-Smith)

Klein et Smith (1951), en utilisant la méthode d'ondes de spin, effectuèrent des intégrations numériques à partir de la formule de Bloch pour l'aimantation

$$M = BN\left[1 - \frac{2(4\pi)}{(2\pi)^2}\int_0^\infty \frac{K^2 dK}{\exp(JK^2/kT) - 1}\right] = BN[1 - 0.13\,(kT/J)^{3/2}]$$

pour des lames allant d'une couche atomique unique jusqu'à 128 couches atomiques, en supposant un système cubique simple. Fig. 2 montre les résultats de ces calculs, notamment, on y voit M/M_0 en fonction T/T_c.

Il résulte des calculs que, à partir d'environ 60 couches atomiques, M/M_0 tombe rapidement en fonction de l'épaisseur, et dépend presque linéairement de T/T_c et non en $(T/T_c)^{3/2}$, comme le supposait Bloch. Donc, il en résulte que même une couche unique peut présenter un caractère ferromagnétique. Comme la théorie des ondes de spin de Bloch n'est valable que pour des températures proches de 0° K (excitation faible des ondes de spin), les résultats de Klein et Smith ne le sont que pour $T/T_c << 1$.

Glass et Klein (1958) ont donné une généralisation de la théorie de Klein et Smith pour un réseau cubique centré et à faces centrées. En utilisant le Hamiltonien de

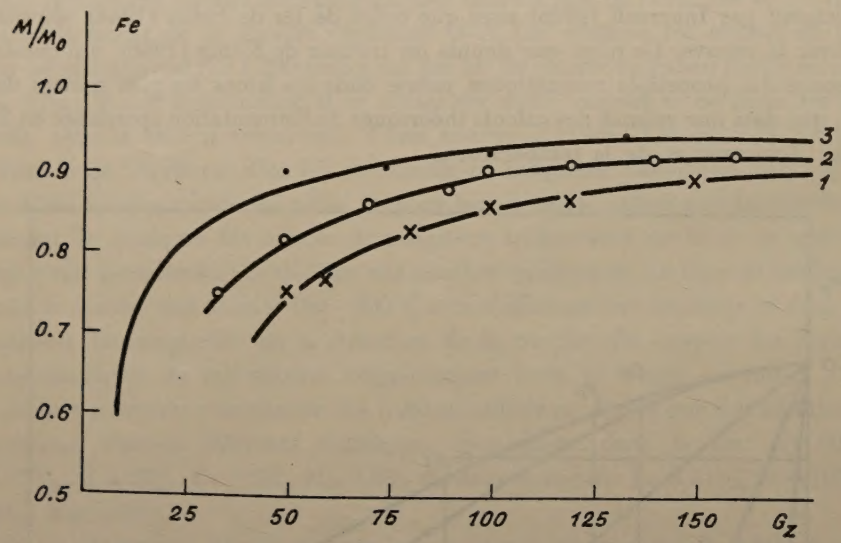

Fig. 3. Aimantation relative M/M_0 en fonction de l'épaisseur G_z pour le fer. La courbe 1 représente les résultats expérimentaux obtenus par l'auteur en utilisant la méthode magnéto-optique (pour une température de 290° K). La courbe 2 provient des mesures magnétométriques de Reincke (1954). La courbe 3 a été calculée théoriquement par Glass-Klein (1958) (pour des températures proches de 0° K).

Holstein-Primakoff et en négligeant l'interaction dipôle-dipôle, Glass et Klein trouvent les valeurs de M/M_0 en fonction de l'épaisseur G_z pour différentes valeurs du paramètre KT/J. Fig. 3 et Fig. 4 montrent les résultats de ces auteurs pour un réseau cubique à faces centrées et pour un réseau cubique centré, respectivement.

Résultats des mesures

L'auteur a fait une étude détaillée de la rotation du plan de polarisation en fonction de l'intensité du champ magnétique dans les lames les plus minces de fer, de nickel, de cobalt et de ferrocobalt. L'aimantation fut déterminée graphiquement d'après les résultats obtenus. Fig. 3 montre en diagramme les résultats obtenus pour le fer;

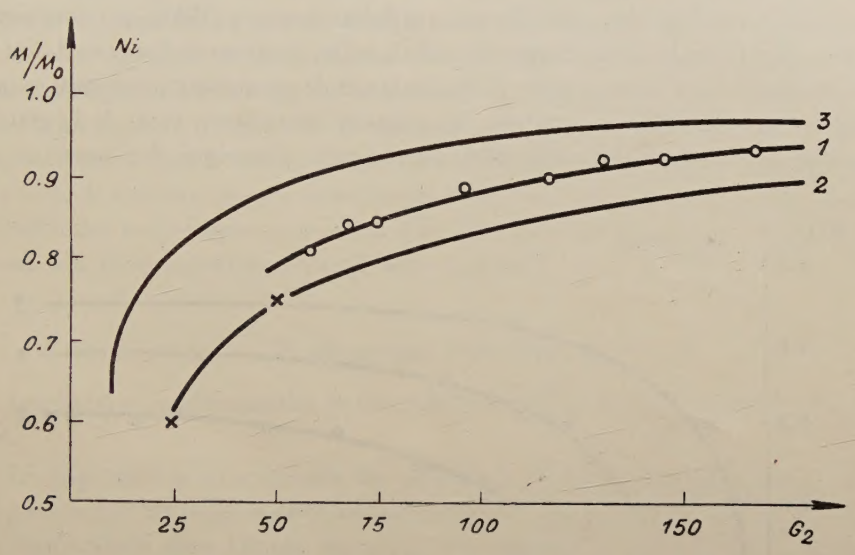

Fig. 4. Aimantation relative M/M_0 en fonction du nombre G_z des couches atomiques, pour le nickel. Courbe 1 — résultats expérimentaux de l'auteur; courbe 2 — résultats expérimentaux de Crittenden-Hoffman (1953); courbe 3 — calculée théoriquement par Glass-Klein (1958).

les valeurs de l'aimantation relative M/M_0 sont portées en fonction de l'épaisseur de la lame G_z donnée par le nombre de couches atomiques du réseau cristallin.

L'allure de l'aimantation relative M/M_0 en fonction de l'épaisseur G_z résultant des expériences faites avec des lames de fer, de nickel, de cobalt et de ferrocobalt prouve que la méthode magnéto-optique se prête, par sa précision, à l'étude des propriétés magnétiques des lames les plus minces des ferromagnétiques ainsi qu'à une étude comparative des théories de Klein-Smith et Glass-Klein. On retiendra, toutefois, que la température des expériences dépasse la limite de valabilité de la théorie des ondes de spin et que les courbes théoriques des diagrammes correspondent à des températures plus basses de deux ordres de grandeur que celle des expériences. La fonction M/M_0 (G_z) pour la structure héxagonale, qui est celle des cristaux du cobalt massif jusqu'à 420° C, n'a pas encore été calculée. On a constaté, toutefois, dans des lames de cobalt, même à la température ambiante, que c'est la structure (hexagonale) cubique à face centrée qui prédomine; c'est celle-ci qui est stable dans le matériel massif au-dessus de 420°C. Ce fait a pu être confirmé par la méthode de diffraction d'électrons. Afin de mieux

accorder les résultats expérimentaux et la théorie, il serait souhaitable de faire les mesures aux environs de 0°K où la théorie des ondes de spin de Bloch est strictement valable. On a trouvé, toutefois, que si l'on veut produire une lame homogène aux propriétés reproductibles, il faut employer un support chauffé (à des températures de 100° à 300°). Jusqu'à présent toutes les tentatives d'obtenir des résultats expérimentaux reproductibles en employant des lames déposées à froid ont échoué. P. ex. des lames métalliques déposées sur support à 77°K ont des propriétés électriques et structurelles différentes, comme l'ont démontré Shurman et Schnackenberg (1942), qui étudiaient la résistance électrique des lames minces de nickel, de fer, de cuivre et d'argent. Ayant subi une condensation aux basses températures, les lames de ces métaux présentent ce qu'on appelle une structure (texture) granulée (les grains de cristallites y étant de la grandeur des particules en suspension colloidale). La résistance électrique des lames de cette

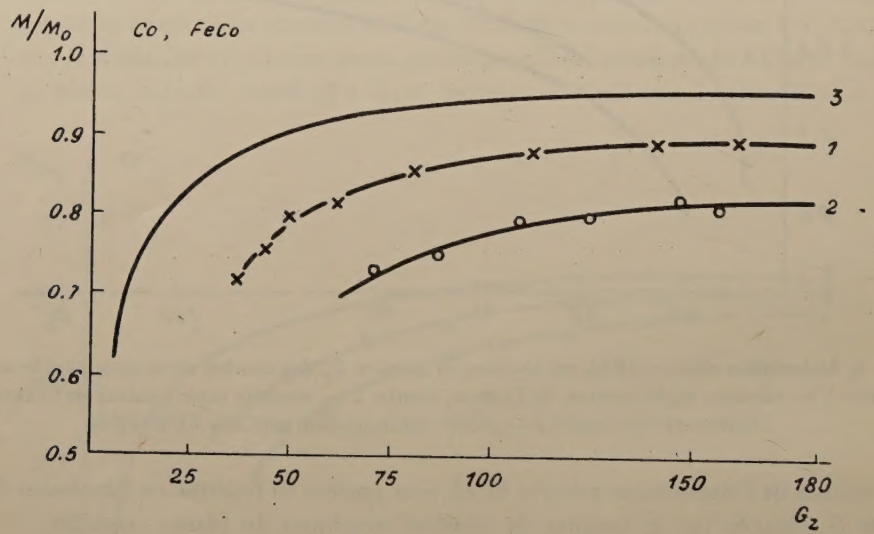

Fig. 5. Aimantation relative M/M_0 en fonction du nombre G_z des couches atomiques. La courbe 1 montre les résultats obtenus par l'auteur pour le cobalt, la courbe 2 — ceux pour un alliage Fe—Co (36% Co). La courbe 3 est celle obtenue à partir des calculs de Glass-Klein pour un réseau cubique à face centrée.

espèce est très grande et atteint sa valeur normale à la suite d'un rechauffement à haute température (recristallisation). La structure granulée se distingue par de nombreuses lacunes et défectuosités dans le réseau, ce qui n'est pas sans influencer les propriétés magnétiques. En effet, Conte et Weil[1], en étudiant les propriétés magnétiques des lames minces de fer et de nickel confectionnées sur support à la température du helium liquide ont constaté qu'une partie de la lame paraît être paramagnétique par suite de la présence de la structure granulée. Le mécanisme déterminant la condition

[1] Colloque International de Magnétisme de Grenoble, 1958.

des atomes en lames minces a pu être suivi en détail grâce à la possibilité d'obtenir le dépôt par sublimation entièrement au-dedans de la camera du microscope ou du diffractographe électronique. Ces études ont confirmé qu'il n'est possible de produire des lames minces de structure homogène qu'en utilisant un support chauffé et un vide élevé. Les électronogrammes des lames déposées aux températures basses sont plutôt caractéristiques de la structure des liquides.

L. Valenta (1957) donne une théorie de l'aimantation spontanée des lames minces ferromagnétiques qui est valable aussi pour des températures élevées, en y introduisant la notion de sous-réseaux de Néel tellement utile dans le cas des antiferromagnétiques. Il admet qu'un sous-réseau est constitué par une couche d'atomes parallèle au support. D'après les résultats de cette théorie, la chute de l'aimantation spontanée aurait lieu à une épaisseur de couche atomique G_z beaucoup plus petite (de 2 à 8), de manière que son contrôle expérimental échappe largement à la précision des méthodes magnétiques courantes. Une telle généralisation de la notion de sous-réseau aux ferromagnétiques paraît assez artificielle.

V. Discussion et conclusions

Les courbes expérimentales et théoriques de M/M_0 en fonction de G_z se ressemblent.

La dispersion assez restreinte des points expérimentaux des courbes 1 (Figs. 3, 4, 5) prouve que la précision de la méthode magnéto-optique est telle que cette méthode peut être utilisée dans l'étude des propriétés magnétiques des lames ferromagnétiques les plus minces. Les courbes théoriques et expérimentales des Figs. 3, 4 et 5 présentent la même allure; les divergences quantitatives s'expliquent par le fait que la température des mesures est de deux ordres de grandeur plus élevée que celle de la valabilité de la théorie des ondes de spin.

La chute de l'aimantation que l'on observe dans les lames minces de fer, de nickel, de cobalt et de ferrocobalt aux épaisseurs d'environ 200 à 500 Å ne résulte pas d'une oxydation des couches, comme l'ont démontré les études structurelles par diffraction des électrons.

Cette chute de l'aimantation est en accord avec les théories de Klein et Smith et de Glass et Klein.

Afin d'éliminer l'oxydation comme cause possible de la diminution de l'aimantation spontanée, diminution constatée dans le cas des lames les plus minces, on a étudié les structures des lames de fer, de cobalt et de nickel en utilisant la méthode de diffraction des électrons. La photographie (Fig. 6) est celle d'un électronogramme d'une lame de fer déposée sur support non chauffé. La diffluence des anneaux permet de conclure à une structure microgranulée non cristallisée. Les photographies (Fig. 7 et Fig. 8) furent obtenues avec une lame de fer déposée sur support chauffé, la première aussitôt après la formation du dépôt, la seconde 24 heures après

sa formation et la dernière (Fig. 9) 48 heures plus tard. Ce n'est que sur la dernière que l'on constate des anneaux très faibles de l'oxyde.

L'auteur remercie le Professeur dr. J. Groniowski, Directeur de l'Institut d'Anatomie Pathologique de l'Académie de Médecine à Poznań, de lui avoir donné la possi-

Fig. 6

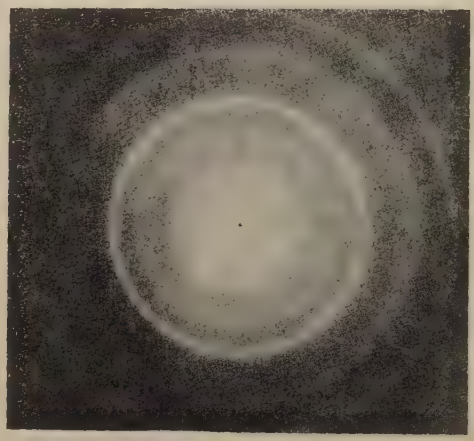

Fig. 7

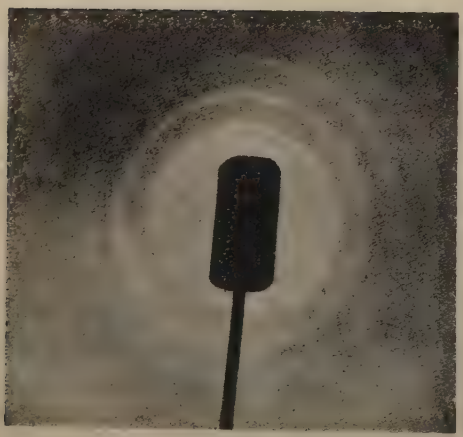

Fig. 8

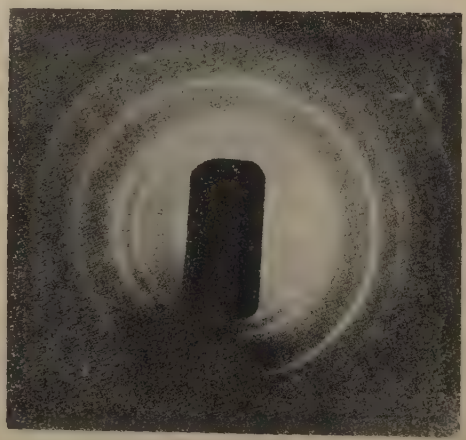

Fig. 9

bilité d'employer le microscope électronique, ainsi que le Docteur W. Djaczenko de lui avoir aidé à effectuer les expériences de diffraction des électrons.

L'auteur exprime ses remerciements chaleureux à M. le Professeur dr. Sz. Szczeniowski pour ses conseils et pour l'aide qu'il lui a bien voulu apporter par ses discussions nombreuses.

L'auteur tient à honorer la mémoire du feu Professeur dr. S. Loria, décédé le 8. Août 1958, sous les auspices duquel le présent travail fut conçu.

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DIFFERENTIAL EQUATIONS IN THE SPINOR SPACE

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The paper is a continuation of some previous work concerned with an attempt to describe physical laws in the spinor space rather than in the Minkowski space. The bilinear connexion of the eight real spinor variables $\mathrm{Re}\, z_\alpha$, $\mathrm{Re}\, z_{\dot\alpha}$, $\mathrm{Im}\, z_\alpha$, $\mathrm{Im}\, z_{\dot\alpha}$ ($\alpha = 1,2$) with the four real vector variables x_μ ($\mu = 1, \ldots 4$) is studied in some detail. It is shown that the remaining four degrees of freedom correspond to four variables φ_μ ($\mu = 1, \ldots 4$) which possess the character of angles or hyperbolic angles. Equations in the spinor space which are covariant with respect to the direct product of two unimodular groups $\mathbf{c} \cdot \mathbf{c}'$ are expressed in terms of the variables x_μ and φ_μ. Thus a correspondence with the conventional treatment in the x_μ — space is established. A provisional discussion of the equations and their solutions is given.

1. Introduction

In a series of papers[1] we have considered some of the consequences of an attempt to describe physical laws in terms of an eight-dimensional spinor space rather than in terms of the conventional four-dimensional Minkowski space. In the present paper we want to give a discussion of some of the covariant differential equations in the spinor space and especially of their correspondence with the conventional Dirac or Klein-Gordon equations in the Minkowski space.

In order to take account of some changes and generalizations of the formalism developed in (I—IV), we begin with a brief summary of some of the results contained in those papers.

If we attempt to locate a physical event in the spinor space of the four complex spinor variables z_α, $z_{\dot\alpha}$ ($\alpha = 1,2$) we immediately see that this space admits a wider group of transformations than the conventional Minkowski space of the four real variables x_μ ($\mu = 1, \ldots 4$). Apart from the unimodular group $\mathbf{c}$,

$$\mathbf{c}: \quad \begin{aligned} z_1' &= \alpha z_1 + \beta z_2, \quad & z_{\dot 1}^{*\prime} &= \alpha z_{\dot 1}^* + \beta z_{\dot 2}^*, \\ z_2' &= \gamma z_1 + \delta z_2, \quad & z_{\dot 2}^{*\prime} &= \gamma z_{\dot 1}^* + \delta z_{\dot 2}^*, \end{aligned} \qquad \alpha\delta - \beta\gamma = 1, \tag{1.1}$$

[1] J. Rzewuski (1958, a, b, c, d,) quoted hereafter as I, II, III and IV respectively

which transforms separately the components of the pairs z_α and $z^\cdot_{\dot\alpha}$, we have namely the possibility of transformations which mix the z_α with the $z^\cdot_{\dot\alpha}$, $z^\star_\alpha$ and $z^{\star\cdot}_{\dot\alpha}$. Among those transformations a particular interest deserve transformations which commute with (1,1) (cf. I, III, IV). One easily shows that the only groups commuting with (1.1) are (i) another six-parameter unimodular group $\boldsymbol{c}'$

$$\boldsymbol{c}': \quad \begin{aligned} z'_\varrho &= az_\varrho + bz^\star_{\dot\varrho}, \\ z^{\star\cdot}_{\dot\varrho} &= cz_\varrho + dz^\star_{\dot\varrho}, \end{aligned} \qquad ad - bc = 1, \qquad (\varrho = 1, 2) \tag{1.2}$$

and (ii) a one-parameter commutative group $\boldsymbol{a}$

$$\boldsymbol{a}: \quad z'_\varrho = e^{i\varphi} z_\varrho, \qquad z^{\star\cdot}_{\dot\varrho} = e^{i\varphi} z^\star_{\dot\varrho}, \qquad (\varrho = 1,2). \tag{1.3}$$

The groups $\boldsymbol{c}$ and $\boldsymbol{c}'$ occur quite symmetrically. One of them, say $\boldsymbol{c}$, is usually connected with rotation of a vector space considered as a representation $D_{\frac{1}{2},\frac{1}{2}}$ of $\boldsymbol{c}$. Thus $\boldsymbol{c}$ corresponds to the ordinary spin. The other groups $\boldsymbol{c}'$ and $\boldsymbol{a}$ may be considered as corresponding to the isotopic spin and to the fermion number respectively.

Due to the commutativity of $\boldsymbol{c}$, $\boldsymbol{c}'$ and $\boldsymbol{a}$ we may construct their direct product

$$\boldsymbol{g} = \boldsymbol{c} \cdot \boldsymbol{c}' \cdot \boldsymbol{a} \tag{1.4}$$

and connect the representations of $\boldsymbol{g}$ with the transformation types of fundamential particles occuring in nature.

Since the representations of $\boldsymbol{a}$ are all one-dimensional, we shall be mainly concerned with representations of $\boldsymbol{c}$ $\boldsymbol{c}'$. The irreducible spaces of these representations are (cf. IV).

$$w^{\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3,\mathcal{J}_4}_{M_1,M_2,M_3,M_4} = \frac{u_1^{\mathcal{J}_1+M_1} u_2^{\mathcal{J}_1-M_1} u_1^{\mathcal{J}_2+M_2} u_2^{\mathcal{J}_2-M_2} v_1^{\mathcal{J}_3+M_3} v_2^{\mathcal{J}_3-M_3} v_1^{\mathcal{J}_4+M_4} v_2^{\mathcal{J}_4-M_4}}{\sqrt{(\mathcal{J}_1+M_1)!\,(\mathcal{J}_1-M_1)!\,(\mathcal{J}_2+M_2)!\,(\mathcal{J}_2-M_2)!\,(\mathcal{J}_3+M_3)!\,(\mathcal{J}_3-M_3)!\,(\mathcal{J}_4+M_4)!\,(\mathcal{J}_4-M_4)!}} \tag{1.5}$$

where u_α, $u^\star_{\dot\alpha}$ transform with respect to $\boldsymbol{g}$ according to the transposed transformation (1.1) and v_α, $v^\star_{\dot\alpha}$ according to the transposed transformation (1.1) with $\alpha, \beta, \gamma, \delta$ replaced by a, b, c, d.

The components of a vector of an arbitrary representation be denoted by

$$f_{\alpha\ldots\beta\dot\gamma\ldots\dot\delta;\varrho\ldots\lambda\dot\sigma\ldots\dot\tau}, \tag{1.6}$$

where the indices before the semicolon transform with respect to $\boldsymbol{g}$ according to (1.1) and the indices after the semicolon according to (1.1) with $\alpha, \beta, \gamma, \delta$ replaced by a, b, c, d.

Representations of the full group $\boldsymbol{g}$ are easily obtained by multiplying (1.5) with the one-dimensional representations of $\boldsymbol{a}$. The transformations obtained in this way cover all the transformation types assumed so far for the fundamental particles including their isotopic spin characterization.

The order of the representations (1.5) or (1.6) is not bounded and, therefore, arbitrary ordinary as well as isotopic spins occur in the formalism. In IV we have pointed out a possibility of restricting the number of representations by considering the vectors u_α, $u_{\dot\alpha}$, v_α, $v_{\dot\alpha}$ as operators satisfying a certain non-commutative algebra. A detailed discussion of this question will be the subject of a separate paper.

Let us mention now briefly another possible approach to the spinor space. In the description of physical laws there occur certain groups of transformations which commute with each other, as e. g. the Lorentz group, the isotopic spin group, the various gauge groups, etc. Denoting the unimodular group homeomorfic with the Lorentz group by $\boldsymbol{c}$, we may write the direct product of all these groups in the form

$$\boldsymbol{g} = \boldsymbol{c} \cdot \boldsymbol{d} \tag{1.7}$$

where $\boldsymbol{d}$ stands for the direct product of all the other groups. It is natural to connect the representations of $\boldsymbol{g}$ with fundamental particles or rather it is natural to choose $\boldsymbol{g}$ in such a way as to obtain all the necessary transformation types. It is quite certain that the unimodular group $\boldsymbol{c}$ must occur in $\boldsymbol{g}$. As to the groups $\boldsymbol{d}$ various assumptions were discussed, as e. g.,

$$\begin{aligned} \boldsymbol{d} &= \boldsymbol{u}' \\ \boldsymbol{d} &= \boldsymbol{u}' \cdot \boldsymbol{a}, \\ \boldsymbol{d} &= \boldsymbol{c}' \cdot \boldsymbol{a}, \\ \boldsymbol{d} &= \boldsymbol{u}' \cdot \boldsymbol{u}'' = \boldsymbol{E}_4. \end{aligned} \tag{1.8}$$

Here $\boldsymbol{u}$ denotes the unitary subgroup of the group $\boldsymbol{c}$ and $\boldsymbol{E}_4$ is the group of rotations of an Euclidean four-dimensional space. The shoice (1.4) corresponds to the third line in (1.8) and contains as particular cases the first and the second line. We have not enough information as yet about the fundamental particles as to decide in favour of one of the possibilities (1.8), or perhaps another possibility not contained in (1.8).

Whatever be the structure of $\boldsymbol{d}$, one always uses in the conventional treatment the variables x_μ of the Minkowski space for the description of physical phenomena. These variables, however, are not true representations of the whole group $\boldsymbol{g}$. So long as the group $\boldsymbol{c}$ (Lorentz group) overshadowed all other groups this choice was natural since the x_μ appear to be the simplest true representation of $\boldsymbol{c}$. At the present stage of the theory of fundamental particles, however, the group $\boldsymbol{d}$ becoming more and more important, we have no reason a priori to give the representation x_μ any preference. In fact by using x_μ we loose from the beginning the possibility of geometrical interpretation of other conservation laws than the conservation of energy-momentum and angular momentum.

Therefore, we have tried to investigate the consequences of the point of view that instead of the vectors x_μ one should use the simplest true representation of the group $\boldsymbol{g}$. In the case $\boldsymbol{g} = \boldsymbol{c} \cdot \boldsymbol{c}' \cdot \boldsymbol{a}$ (the case $\boldsymbol{g} = \boldsymbol{c} \cdot \boldsymbol{E}_4$ will be discussed elsewhere)

the simplest true representation of $\boldsymbol{c} \cdot \boldsymbol{c}'$ is given (cf. (1.6)) by the 16 complex variables

$$z_{\alpha;\beta}, \quad z_{\dot{\alpha};\dot{\beta}}, \quad z_{\alpha;\dot{\beta}}, \quad z_{\dot{\alpha};\beta}. \tag{1.9}$$

To reduce the number we add certain reality conditions

$$z_{\dot{\alpha};\dot{\beta}} = z^{*}_{\alpha;\beta}, \quad z_{\alpha;\dot{\beta}} = z^{*}_{\dot{\alpha};\beta}. \tag{1.10}$$

Each of the sets $z_{\alpha;\beta}$ and $z_{\alpha;\dot{\beta}}$ is a true representation of $\boldsymbol{c} \cdot \boldsymbol{c}'$. The transformations of $\boldsymbol{g}$ do not mix the two sets (representation (1.9) is reducible and splits into four irreducible representations $z_{\alpha;\beta}$, $z_{\dot{\alpha};\dot{\beta}}$, $z_{\alpha;\dot{\beta}}$ and $z_{\dot{\alpha};\beta}$). If $z_{\alpha;\beta}$ is transformed by a particular element $c \cdot c'$ of $\boldsymbol{c} \cdot \boldsymbol{c}'$, then $z_{\alpha;\dot{\beta}}$ is transformed by the element $c \cdot c'^{*}$ of $\boldsymbol{c} \cdot \boldsymbol{c}'$, where c'^{*} is the complex conjugate of c'. But $\boldsymbol{c}$ is independent of $\boldsymbol{c}'$, so we may consider $z_{\alpha;\beta}$ and $z_{\alpha;\dot{\beta}}$ as not essentially different (cf. however the considerations connected with inversions at the end of this section).

The variables $z_{\alpha;\beta}$ (or $z_{\alpha;\dot{\beta}}$) are easily connected with the variables z_{α}, $z_{\dot{\alpha}}$ (without semicolon) introduced by the transformation properties (1.1—3)). We may put namely, in accordance with these transformation properties,

$$\begin{aligned} z_1 &= z_{1;1}, \quad & z_2 &= z_{2;1}, \\ z_{\dot{1}} &= z_{\dot{1};\dot{2}}, \quad & z_{\dot{2}} &= z_{\dot{2};\dot{2}}. \end{aligned} \tag{1.11}$$

The group $\boldsymbol{g} = \boldsymbol{c} \cdot \boldsymbol{c}' \cdot \boldsymbol{a}$ is, therefore, distinguished as the only group which may be defined on a four-dimensional complex space and which contains $\boldsymbol{c}$ as a factor.

The quantity

$$\begin{aligned} z_{\alpha;\beta} z^{\alpha;\beta} &\equiv 2 (z_{1;1} z_{2;2} - z_{1;2} z_{2;1}) \\ &\equiv 2 (z_1 z^{*}_2 - z_2 z^{*}_1) \equiv 2 z_{\alpha} z^{\dot{\alpha}*} \end{aligned} \tag{1.12}$$

is invariant with respect to $\boldsymbol{c} \cdot \boldsymbol{c}'$. A transformation of $\boldsymbol{a}$ multiplies (1.12) with a phase factor $e^{2i\varphi}$. Therefore, the only quantity invariant with respect to the full group $\boldsymbol{g}$ is

$$|z_{\alpha;\beta} z^{\alpha;\beta}| \equiv 2 |z^{*}_{\alpha} z^{\dot{\alpha}}|. \tag{1.13}$$

Here the usual convention about rising and lowering the spinor indices is adopted as well for the notation with semicolon as for notation without semicolon.

To establish correspondence with the usual treatment, we have to express the vector coordinates

$$\begin{aligned} z_{1\dot{1};} &= x_3 - x_0, \quad & z_{1\dot{2};} &= x_1 - i x_2, \\ z_{2\dot{1};} &= x_1 + i x_2, \quad & z_{2\dot{2};} &= -x_3 - x_0, \end{aligned} \tag{1.14}$$

by means of $z_{\alpha;\beta}$. They must be scalars with respect to $\boldsymbol{c}'$ (we shall call such quantities c-vectors — c'-scalars). The simplest connexion is the bilinear connexion discussed in IV

$$z_{\alpha\dot{\beta};} = z_{\alpha;\varrho} z_{\dot{\beta};\dot{\lambda}} \varepsilon^{\dot{\varrho}\dot{\lambda}}, \tag{1.15}$$

in which there appears automatically a constant c-scalar — c'-vector $\varepsilon^{;\varrho\dot{\lambda}}$. Using the formulae

$$\begin{aligned} z_{\alpha;\varrho}\, z^{\alpha;\sigma} &= \tfrac{1}{2}\,\delta^{;\sigma}_{;\varrho}\, z_{\alpha;\beta}\, z^{\alpha;\beta} \equiv \delta^{;\sigma}_{;\varrho}\, z_{\alpha}\, z^{\dot{\alpha}\star}, \\ z_{\alpha;\varrho}\, z^{\beta;\varrho} &= \tfrac{1}{2}\,\delta^{\beta;}_{\alpha;}\, z_{\varrho;\sigma}\, z^{\varrho;\sigma} \equiv \delta^{\beta;}_{\alpha;}\, z_{\varrho}\, z^{\dot{\varrho}\star}, \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} \varepsilon_{;1\dot{1}} &= \varepsilon_3 - \varepsilon_0, \qquad \varepsilon_{;1\dot{2}} = \varepsilon_1 - i\varepsilon_2, \\ \varepsilon_{;2\dot{1}} &= \varepsilon_1 + i\varepsilon_2, \qquad \varepsilon_{;2\dot{2}} = -\varepsilon_3 - \varepsilon_0, \end{aligned} \tag{1.17}$$

we easily derive

$$x_\mu^2 = -\tfrac{1}{2}\, z_{\alpha\dot{\beta};}\, z^{\alpha\dot{\beta};} = \varepsilon_\mu^2\, |z_\alpha^\star\, z^{\dot{\alpha}}|^2. \tag{1.18}$$

It is seen further that the $z_{\alpha\dot{\beta};}$ defined by (1.15) are invariant also with respect to $\boldsymbol{a}$.

Les us discuss briefly the question of inversions which is related closely to the bilinear form (1.15). It may be seen easily that inversions of space $I_x^{\rightarrow}$ and time I_t

$$z'_{\alpha\dot{\beta};} = \pm\, z^{\beta\dot{\alpha};}, \tag{1.19}$$

may be realized by the following transformations of the $z_{\alpha;\beta}$

$$z'_{\alpha;\beta} = a z^{\dot{\alpha};}{}_{;\dot{\beta}}, \qquad |a| = 1, \tag{1.20}$$

if at the same time we perform the following transformation of the $\varepsilon^{;\alpha\dot{\beta}}$

$$\varepsilon^{;\alpha\dot{\beta}'} = \pm\, \varepsilon^{;\beta\dot{\alpha}}. \tag{1.21}$$

Transformation (1.21) with the $+$ sign is the inversion of the second axis in the c'-vector space corresponding to the isotopic group $\boldsymbol{c'}$, and is equivalent with charge conjugation C. Transformation (1.21) with the — sign is C times a strong reflection I_{ε_μ} ($\varepsilon'_\mu = -\varepsilon_\mu$) in the c'-vector space. Thus we obtain the result that in the spinor space of the variables $z_{\alpha;\beta}$ neither space and time inversions (P and T respectively) nor charge conjugation (C) may be realized separately by means of transformations which do not depend on the constant vector $\varepsilon^{;\varrho\dot{\lambda}}$. The transformations (1.19—21) are just PC and $TCI_{\varepsilon_\mu} = T_W I_{\varepsilon_\mu}$, where T_W is the time inversion in Wigner's sense.

To obtain P and T separately we have to use transformations which contain the constant c'-vector $\varepsilon_{;\alpha\dot{\beta}}$ and even then we obtain only partial results. In fact, one easily verifies by means of the relations

$$\varepsilon^{;\sigma\dot{\lambda}}\, \varepsilon_{;\varrho\dot{\lambda}} = -\varepsilon_\mu^2\, \delta^{;\sigma}_{;\varrho} \tag{1.22}$$

that for space-like vectors ($\varepsilon_\mu^2 = 1$, $x_\mu^2 > 0$)

$$\begin{aligned} &I_x^{\rightarrow}: && z'_{\alpha;\beta} = a z^{\dot{\alpha};\dot{\lambda}}\, \varepsilon_{;\dot{\lambda}\beta}, && \varepsilon'_{;\alpha\dot{\beta}} = -\varepsilon_{;\alpha\dot{\beta}}, && |a| = 1, \\ &I_t: && && \varepsilon'_{;\alpha\dot{\beta}} = \varepsilon_{;\alpha\dot{\beta}}, \end{aligned} \tag{1.23}$$

and for time-like vectors ($\varepsilon_\mu^2 = -1$, $x_\mu^2 < 0$)

$$\begin{matrix} I_x^+: \\ I_t: \end{matrix} \quad z'_{\alpha;\beta} = a z^{\dot{\alpha};\dot{\lambda}} \varepsilon_{;\dot{\lambda}\beta}, \qquad \begin{matrix} \varepsilon'_{;\alpha\dot{\beta}} = \varepsilon_{;\alpha\dot{\beta}}, \\ \varepsilon'_{;\alpha\dot{\beta}} = -\varepsilon_{;\alpha\dot{\beta}}, \end{matrix} \qquad |a| = 1. \tag{1.24}$$

Thus for space-like vectors we may obtain separately only T. P is connected with strong reflection of the c'-vector ε_μ: $\varepsilon'_\mu = -\varepsilon_\mu$. For time-like vectors the situation is reversed. One obtains separately only P. T is connected with the strong reflection $\varepsilon'_\mu = -\varepsilon_\mu$.

It may be mentioned that in order to obtain P and T separately we have to double the number of variables (a situation analogous to that which occurs for ordinary spinors when dotted indices are introduced to represent the full Lorentz group including inversions) and to introduce besides $z_{\alpha;\beta}$ the set $z_{\dot{\alpha};\dot{\beta}}$. In this case the connection with a real c-vector may be written

$$z_{\alpha\dot{\beta};} = i(z_{\alpha;\gamma} z_{\dot{\beta};}^{;\gamma} + z_{\alpha;\dot{\gamma}} z_{\dot{\beta};}^{;\dot{\gamma}}) \tag{1.25}$$

without the help of a constant c'-vector. Inversions of space and time (1.19) are now easily obtained by means of the transformations

$$z'_{\alpha;\gamma} = z^{\dot{\alpha};}_{;\gamma}, \qquad z'_{\dot{\alpha};\gamma} = \mp z^{\alpha;}_{;\gamma}. \tag{1.26}$$

The impossibility to obtain P and T separately in the framework of the spinor space with four complex variables $z_{\alpha;\beta}$ might be considered as a clue for understanding the parity non-conservation. On the other hand, since parity non-conservation cannot be understood on the basis of Minkowski's geometry, this may be considered as an argument in favour of introducing the (four-dimensional complex) spinor space. We hope to be able to give a more detailed discussion of these questions elsewhere.

Having thus reviewed all those formulae which will be necessary in the understanding of further developements we may go over to the main question of this paper, namely to the study of the bilinear connexion (1.15) and to the discussion of covariant differential equations. Let us start with some general remarks about the latter.

2. *Differential Equations*

The general connection of differential equations in the spinor space and in the vector space was briefly studied in II on the basis of a particular case of formula (1.15) (cf. however also IV). In this section we shall give a somewhat more advanced treatement of this problem based on formula (1.15).

Let us consider the possible linear differential equations of first or second order respectively, covariant with respect to the group $\boldsymbol{c} \cdot \boldsymbol{c}'$.

For a scalar f we have the two second order equations

$$\frac{\partial^2 f}{\partial z_{\alpha;\beta} \partial z^{\alpha;\beta}} = af \quad \text{or} \quad \frac{\partial^2 f}{\partial z_{\dot{\alpha};\dot{\beta}} \partial z^{\dot{\alpha};\dot{\beta}}} = af. \tag{2.1}$$

For a c-spinor — c'-scalar $f_{\alpha;}$, $f_{\dot\alpha;}$ we have the two alternatives: (i) the two-component first order equations

$$\frac{\partial f_{\alpha;}}{\partial z_{\alpha;\varrho}} = t^{\alpha;\varrho} f_{\alpha;}, \qquad \frac{\partial f_{\dot\alpha;}}{\partial z_{\dot\alpha;\dot\varrho}} = t^{\dot\alpha;\dot\varrho} f_{\dot\alpha;}, \tag{2.2}$$

or

$$t^{\beta;\gamma} \frac{\partial f_{\beta;}}{\partial z^{\alpha;\gamma}} = f_{\alpha;}, \qquad t^{\dot\beta;\dot\gamma} \frac{\partial f_{\dot\beta;}}{\partial z^{\dot\alpha;\dot\gamma}} = f_{\dot\alpha;} \tag{2.2'}$$

and (ii) the four-component first order equations

$$\frac{\partial f_{\alpha;}}{\partial z_{\alpha;\varrho}} = t^{\dot\alpha;\varrho} f_{\dot\alpha;}, \qquad \frac{\partial f_{\dot\alpha;}}{\partial z_{\dot\alpha;\dot\varrho}} = t^{\alpha;\dot\varrho} f_{\alpha;}, \tag{2.3}$$

or

$$t^{\beta;\dot\gamma} \frac{\partial f_{\beta;}}{\partial z^{\alpha;\dot\gamma}} = f_{\dot\alpha;}, \qquad t^{\dot\beta;\gamma} \frac{\partial f_{\dot\beta;}}{\partial z^{\alpha;\gamma}} = f_{\alpha;}. \tag{2.3'}$$

Similarly for a c-scalar — c'-spinor $f_{;\alpha}$, $f_{;\dot\alpha}$ we have equations of the type

$$\frac{\partial f_{;\alpha}}{\partial z_{\varrho;\alpha}} = t^{\varrho;\alpha} f_{;\alpha}, \qquad \frac{\partial f_{;\dot\alpha}}{\partial z_{\dot\varrho;\dot\alpha}} = t^{\dot\varrho;\dot\alpha} f_{;\dot\alpha} \tag{2.4}$$

$$\frac{\partial f_{;\alpha}}{\delta z_{\varrho;\alpha}} = t^{\varrho;\dot\alpha} f_{;\dot\alpha}, \qquad \frac{\partial f_{;\dot\alpha}}{\partial z_{\dot\varrho;\dot\alpha}} = t^{\dot\varrho;\alpha} f_{;\alpha}, \text{ etc.} \tag{2.5}$$

We may proceed in this way, considering equations for a c-spinor — c'-spinor, c-spinor — c'-vector etc. In this paper, however, we shall be concerned only with equations (2.1—3). It may be mentioned in this connection that equations (2.2—3) may be used also for fields with arbitrary transformation character after the semicolon, with the understanding that the additional indices are the same on both sides of the equation. The quantities a, $t^{\alpha;\varrho}$, $t^{\dot\alpha;\dot\varrho}$, $t^{\alpha;\dot\varrho}$, $t^{\dot\alpha;\varrho}$, are so far arbitrary functions of $z_{\alpha;\varrho}$ possessing the appropriate transformation character.

To investigate the connexion of equations (2.1—3) with the Klein-Gordon or Dirac eqations respectively, we shall need the connexion between the derivatives with respect to the variables of the spinor space $\frac{\partial}{\partial z_{\alpha;\beta}}$, $\frac{\partial}{\partial z_{\dot\alpha;\dot\beta}}$ and the derivatives with respect to the variables of the vector space $\frac{\partial}{\partial z_{\alpha\dot\beta;}}$. Consider an arbitrary function which depends on the variables $z_{\alpha;\beta}$, $z_{\dot\alpha;\dot\beta}$ as well explicitly as implicitly by means of the $z_{\alpha\dot\beta;}$ of formula (1.15):

$$f = f[z_{\alpha\dot\beta;}(z_{\varrho;\lambda},\ z_{\dot\varrho;\dot\lambda});\ z_{\varrho;\lambda},\ z_{\dot\varrho;\dot\lambda}]. \tag{2.6}$$

We have

$$\frac{\partial f}{\partial z_{\alpha;\beta}} = \frac{\partial z_{\varrho\dot{\lambda};}}{\partial z_{\alpha;\beta}} \frac{\partial f}{\partial z_{\varrho\dot{\lambda};}} + \frac{\partial f}{\partial z'_{\alpha;\beta}}, \tag{2.7}$$

where $\frac{\partial}{\partial z'_{\alpha;\beta}}$ denotes the derivatives with respect to the variables $z_{\alpha;\beta}$ occuring explicitly in (2.6). Due to (1.15)

$$\frac{\partial z_{\varrho\dot{\lambda};}}{\partial z_{\alpha;\beta}} = \delta^{\alpha;}_{\varrho;} z_{\dot{\lambda};\dot{\tau}} \varepsilon^{\beta\dot{\tau}}. \tag{2.8}$$

Introducing (2.8) into (2.7) we obtain

$$\frac{\partial}{\partial z_{\alpha;\beta}} = z_{\dot{\lambda};\dot{\tau}} \varepsilon^{\beta\dot{\tau}} \frac{\partial}{\partial z_{\alpha\dot{\lambda};}} + \frac{\partial}{\partial z'_{\alpha;\beta}}. \tag{2.9}$$

With help of relations (1.16) and (1.22) these equations may be solved with respect to $\frac{\partial}{\partial z_{\alpha\dot{\lambda};}}$

$$z^{\dot{\nu};\dot{\mu}} \varepsilon_{,\beta\dot{\mu}} \left(\frac{\partial}{\partial z_{\alpha;\beta}} - \frac{\partial}{\partial z'_{\alpha;\beta}} \right) = -\varepsilon_\mu^2 z_\varrho^* z^{\dot{\varrho}} \frac{\partial}{\partial z_{\alpha\dot{\nu};}}. \tag{2.10}$$

If $z_\varrho^* z^{\dot{\varrho}} \neq 0$, i. e. outside the light-cone (cf. (1.18)), we may rewrite this formula in the form

$$\frac{\partial}{\partial z_{\alpha\dot{\beta};}} = \frac{-1}{\varepsilon_\mu^2 z_\sigma^* z^{\dot{\sigma}}} \cdot z^{\dot{\beta};\dot{\varrho}} \varepsilon_{;\gamma\dot{\varrho}} \left(\frac{\partial}{\partial z_{\alpha;\gamma}} - \frac{\partial}{\partial z'_{\alpha;\gamma}} \right). \tag{2.11}$$

Connexions (2.9) and (2.11) enable us to establish the correspondence between covariant differential equations in the spinor space and in the c-vector space.

Let us consider first equations (2.2). Introducing these equations into (2.11) and into the complex conjugate relation respectively, we obtain

$$\begin{aligned} \frac{\partial f_{\alpha;}}{\partial z_{\alpha\dot{\beta};}} &= \frac{-1}{\varepsilon_\mu^2 z_\sigma^* z^{\dot{\sigma}}} \cdot z^{\dot{\beta};\dot{\varrho}} \varepsilon_{;\gamma\dot{\varrho}} \left(t^{\alpha;\gamma} f_{\alpha;} - \frac{\partial f_{\alpha;}}{\partial z'_{\alpha;\gamma}} \right), \\ \frac{\partial f^{\dot{\alpha}}_{;}}{\partial z_{\dot{\alpha}\beta;}} &= \frac{-1}{\varepsilon_\mu^2 z_\sigma z^{\dot{\sigma}^*}} \cdot z^{\beta;\varrho} \varepsilon_{;\dot{\gamma}\varrho} \left(t^{\dot{\alpha};\dot{\gamma}} f^{\dot{\alpha}}_{;} - \frac{\partial f^{\dot{\alpha}}_{;}}{\partial z'_{\dot{\alpha};\dot{\gamma}}} \right), \end{aligned} \tag{2.12}$$

where the left hand sides are the Dirac differential operators in explicit form (cf. (1.14)). In the case when $f_{\alpha;}$ and $f^{\dot{\alpha}}_{;}$ do not depend explicitly on $z_{\alpha;\beta}$, the derivatives $\frac{\partial f_{\alpha;}}{\partial z'_{\alpha;\gamma}}$ and $\frac{\partial f^{\dot{\alpha}}_{;}}{\partial z'_{\dot{\alpha};\dot{\gamma}}}$ vanish and we obtain from (2.12) two-component Dirac equations with vanishing rest-mass and with certain additional terms determined by $t^{\alpha;\gamma}$ and $t^{\dot{\alpha};\dot{\gamma}}$. Correspondence with two-component mass-less Dirac equations is obtained, therefore, in the case when there is no explicit dependence on $z_{\alpha;\beta}$ and when besides $t^{\alpha;\gamma} = t^{\dot{\alpha};\dot{\gamma}} = 0$.

Consider now equations (2.3). Introducing these equations into (2.11) and into the complex conjugate to (2.11), we obtain respectively

$$\frac{\partial f_{\alpha;}}{\partial z_{\alpha\dot\beta;}} = \frac{-1}{\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}} \cdot z^{\dot\beta;\dot\varrho}\, \varepsilon_{;\gamma\dot\varrho} \left(t^{\dot\alpha;\gamma} f_{\dot\alpha;} - \frac{\partial f_{\alpha;}}{\partial z'_{\dot\alpha;\gamma}} \right),$$

$$\frac{\partial f_{\dot\alpha;}}{\partial z_{\dot\alpha\beta;}} = \frac{-1}{\varepsilon_\mu^2 z_\sigma z^{\dot\sigma\star}} \cdot z^{\beta;\varrho}\, \varepsilon_{;\dot\gamma\varrho} \left(t^{\alpha;\dot\gamma} f_{\alpha;} - \frac{\partial f_{\dot\alpha;}}{\partial z'_{\dot\alpha;\dot\gamma}} \right). \tag{2.13}$$

The situation concerning the correspondence with two-component mass-less Dirac equations is here the same as for equations (2.12). However, there exists one choice of $t^{\dot\alpha;\gamma}$ which leads to a correspondence with four-component Dirac equations with non-vanishing rest-mass. Indeed, taking

$$\begin{aligned} t^{\dot\alpha;\gamma} &= a z^{\dot\alpha;}_{;\tau}\, \varepsilon^{;\gamma\tau}, \\ t^{\alpha;\dot\gamma} &= b z^{\alpha;}_{;\tau}\, \varepsilon^{;\dot\gamma\tau}, \end{aligned} \tag{2.14}$$

we obtain from (2.13(due to (1.16) and (1.22)

$$\frac{\partial f_{\alpha;}}{\partial z_{\alpha\dot\beta;}} = -a f^{\dot\beta;} + \frac{1}{\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}} \cdot z^{\dot\beta;\dot\varrho}\, \varepsilon_{;\gamma\dot\varrho} \frac{\partial f_{\alpha;}}{\partial z'_{\alpha;\gamma}},$$

$$\frac{\partial f_{\dot\alpha;}}{\partial z_{\dot\alpha\beta;}} = -b f^{\beta;} + \frac{1}{\varepsilon_\mu^2 z_\sigma z^{\dot\sigma\star}} \cdot z^{\beta;\varrho}\, \varepsilon_{;\dot\gamma\varrho} \frac{\partial f_{\dot\alpha;}}{\partial z'_{\dot\alpha;\dot\gamma}}. \tag{2.15}$$

If we consider a and b as constant factors, equations (2.15) go over into the full Dirac equations when the explicit dependence of $f_{\alpha;}$, $f_{\dot\alpha;}$ on $z_{\alpha;\beta}$ is disregarded.

We may also consider the set consisting of the first equation (2.3) and the second of equations (2.3′). Due to (2.11) we get in this case

$$\frac{\partial f_{\alpha;}}{\partial z_{\alpha\dot\beta;}} = \frac{-1}{\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}} \cdot z^{\dot\beta;\dot\varrho}\, \varepsilon_{;\gamma\dot\varrho} \left(t^{\dot\alpha;\gamma} f_{\dot\alpha;} - \frac{\partial f_{\alpha;}}{\partial z'_{\dot\alpha;\gamma}} \right),$$

$$\frac{\partial f^{\dot\beta;}}{\partial z^{\alpha\dot\beta;}} = \frac{-1}{\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}} \cdot z_{\dot\beta;\dot\varrho}\, \varepsilon^{;\gamma\dot\varrho} \left(\frac{\partial f^{\dot\beta;}}{\partial z^{\alpha;\gamma}} - \frac{\partial f^{\dot\beta;}}{\partial z^{\alpha;\gamma'}} \right). \tag{2.13′}$$

Correspondence with Dirac equations is obtained by taking in the first equation (2.3) $t^{\dot\alpha;\gamma}$ given by (2.14) and in the second of equations (2.3′) $t^{\dot\beta;\gamma}$ given by

$$b t_{\dot\beta;}^{\;;\gamma} = \frac{1}{\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}} \cdot z_{\dot\beta;\dot\varrho}\, \varepsilon^{;\gamma\dot\varrho}. \tag{2.14′}$$

Equations (2.13′) go over in this case into

$$\frac{\partial f_{\alpha;}}{\partial z_{\alpha\dot\beta;}} = -a f^{\dot\beta;} + \frac{1}{\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}} \cdot z^{\dot\beta;\dot\varrho}\, \varepsilon_{;\gamma\dot\varrho} \frac{\partial f_{\alpha;}}{\partial z'_{\alpha;\gamma}},$$

$$\frac{\partial f^{\dot\beta;}}{\partial z^{\alpha\dot\beta;}} = b f_{\alpha;} + \frac{1}{\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}} \cdot z_{\dot\beta;\dot\varrho}\, \varepsilon^{;\gamma\dot\varrho} \frac{\partial f^{\dot\beta;}}{\partial z^{\alpha;\gamma'}}. \tag{2.15′}$$

We shall consider the problem of correspondence more closely in the next sections after having established the form of the operators containing the explicit derivatives with respect to $z_{\alpha;\beta}$. Now let us go over finally to the consideration of the second order operator (2.1). From (2.7) we get by a second differentiation

$$\frac{\partial^2 f}{\partial z_{\alpha;\beta}\,\partial z^{\alpha;\beta}} = \frac{\partial^2 z_{\varrho\dot{\lambda};}}{\partial z_{\alpha;\beta}\,\partial z^{\alpha;\beta}}\,\frac{\partial f}{\partial z_{\varrho\dot{\lambda};}} + \frac{\partial z^{\varrho\dot{\lambda};}}{\partial z^{\alpha;\beta}}\,\frac{\partial z_{\mu\dot{\nu};}}{\partial z_{\alpha;\beta}}\,\frac{\partial^2 f}{\partial z_{\mu\dot{\nu};}\,\partial z^{\varrho\dot{\lambda};}} +$$

$$+ \frac{\partial z^{\varrho\dot{\lambda};}}{\partial z^{\alpha;\beta}}\,\frac{\partial^2 f}{\partial z'_{\alpha;\beta}\,\partial z^{\varrho\dot{\lambda};}} + \frac{\partial z_{\mu\dot{\nu};}}{\partial z_{\alpha;\beta}}\,\frac{\partial^2 f}{\partial z_{\mu\dot{\nu};}\,\partial z^{\alpha;\beta'}} + \frac{\partial^2 f}{\partial z'_{\alpha;\beta}\,\partial z^{\alpha;\beta'}}. \tag{2.16}$$

Due to (2.8) and the corresponding formula

$$\frac{\partial z^{\varrho\dot{\lambda};}}{\partial z^{\alpha;\beta}} = \delta^{\varrho;}_{\alpha;}\, z^{\dot{\lambda};\dot{\tau}}\,\varepsilon_{;\beta\dot{\tau}}, \tag{2.17}$$

we get

$$\frac{\partial^2 z_{\varrho\dot{\lambda};}}{\partial z_{\alpha;\beta}\,\partial z^{\alpha;\beta}} = 0, \qquad \frac{\partial z^{\varrho\dot{\lambda};}}{\partial z^{\alpha;\beta}}\,\frac{\partial z_{\mu\dot{\nu};}}{\partial z_{\alpha;\beta}} = -\,\varepsilon_{\mu}^{2}\,\delta^{\varrho;}_{\mu;}\,\delta^{\dot{\lambda};}_{\dot{\nu};}\,z^{\star}_{\sigma}\,z^{\dot{\sigma}},$$

and, by virtue of (2.1),

$$\frac{\partial^2 f}{\partial z_{\alpha;\beta}\,\partial z^{\alpha;\beta}} = -\,\varepsilon_{\mu}^{2}\,z^{\star}_{\sigma}\,z^{\dot{\sigma}}\,\frac{\partial^2 f}{\partial z_{\varrho\dot{\lambda};}\,\partial z^{\varrho\dot{\lambda};}} + z^{\dot{\lambda};\dot{\tau}}\,\varepsilon_{;\beta\dot{\tau}}\,\frac{\partial^2 f}{\partial z'_{\alpha;\beta}\,\partial z^{\alpha\dot{\lambda};}} +$$

$$+ z_{\dot{\lambda};\dot{\tau}}\,\varepsilon^{;\beta\dot{\tau}}\,\frac{\partial^2 f}{\partial z_{\alpha\dot{\lambda};}\,\partial z^{\alpha;\beta'}} + \frac{\partial^2 f}{\partial z'_{\alpha;\beta}\,\partial z^{\alpha;\beta'}} = af. \tag{2.18}$$

Since $\dfrac{\partial^2}{\partial z_{\varrho\dot{\lambda};}\,\partial z^{\varrho\dot{\lambda};}}$ is proportional to the d'Alembertian, we get correspondence with the Klein-Gordon equation by neglecting the explicit dependence of f on $z_{\alpha;\beta}$ and by putting $a \sim z^{\star}_{\sigma}\,z^{\dot{\sigma}}$.

In this paper we shall be mainly concerned with equations (2.15), (2.15′) and (2.18). It will prove convenient to rewrite (2.15) and (2.15′) in more customary notation.

For this purpose we put

$$p^{\alpha\dot{\beta};} = \frac{1}{\varepsilon_{\mu}^{2}\,z^{\star}_{\sigma}\,z^{\dot{\sigma}}}\cdot z^{\dot{\beta};\dot{\varrho}}\,\varepsilon_{;\gamma\dot{\varrho}}\,\frac{\partial}{\partial z'_{\alpha;\gamma}}, \tag{2.19}$$

and

$$\partial^{\alpha\dot{\beta};} = \frac{\partial}{\partial z_{\alpha\dot{\beta};}}, \tag{2.20}$$

Equations (2.15) and (2.15′) may now be written in the form

$$\begin{aligned} (\partial^{\dot{\beta}\alpha;} - p^{\dot{\beta}\alpha;})\,f_{\alpha;} &= -\,af^{\dot{\beta};}, \\ (\partial_{\beta\dot{\alpha};} - p^{\star}_{\beta\dot{\alpha};})\,f^{\dot{\alpha};} &= \quad bf_{\beta;}, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned}(\partial^{\dot\beta\alpha_i} - p^{\dot\beta\alpha_i}) f_{\alpha_i} &= -af^{\dot\beta_i},\\ (\partial_{\alpha\dot\beta_i} - p_{\alpha\dot\beta_i}) f^{\dot\beta_i} &= \quad bf_{\alpha_i}.\end{aligned} \tag{2.21'}$$

Due to (1.14) we have $\left(\partial_\mu = \frac{\partial}{\partial x_\mu}\right)$

$$(\partial^{\dot\beta\alpha_i}) = \frac{1}{2}\begin{pmatrix}\partial_3 - \partial_0 & \partial_1 - i\partial_2\\ \partial_1 + i\partial_2 & -\partial_3 - \partial_0\end{pmatrix} = \frac{1}{2}(\sigma_i\partial_i - \sigma_0\partial_0), \tag{2.22}$$

where σ_i are the conventional Pauli matrices

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{2.23}$$

We define also a complex vector operator p_μ by means of the relations

$$(p^{\dot\beta\alpha_i}) = \frac{1}{2}\begin{pmatrix}p_3 + p_0 & p_1 - ip_2\\ p_1 + ip_2 & -p_3 + p_0\end{pmatrix} = \frac{1}{2}(\sigma_i p_i + \sigma_0 p_0) \tag{2.24}$$

Denoting

$$\varphi = \begin{pmatrix}f_{1_i}\\ f_{2_i}\end{pmatrix}, \quad \chi = \begin{pmatrix}f^{\dot 1_i}\\ f^{\dot 2_i}\end{pmatrix}, \tag{2.25}$$

we may write (2.21) and (2.21') in operator form

$$\begin{aligned}[\sigma_i(\partial_i - p_i) - \sigma_0(\partial_0 + p_0)]\,\varphi &= -2a\,\chi,\\ [\sigma_i(\partial_i - p_i^*) + \sigma_0(\partial_0 + p_0^*)]\,\chi &= -2b\,\varphi,\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}[\sigma_i(\partial_i - p_i) - \sigma_0(\partial_0 + p_0)]\,\varphi &= -2a\,\chi,\\ [\sigma_i(\partial_i - p_i) + \sigma_0(\partial_0 + p_0)]\,\chi &= -2b\,\varphi.\end{aligned} \tag{2.26'}$$

Introducing the four-dimensional γ_μ – matrices

$$\gamma = -\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}, \quad \gamma_4 = \begin{pmatrix}0 & -i\sigma_0\\ i\sigma_0 & 0\end{pmatrix}, \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix}, \tag{2.27}$$

and the four-spinor

$$\varphi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}, \tag{2.28}$$

splitting p_μ into real and imaginary parts

$$p_\mu = q_\mu + ir_\mu, \quad (\mu = 0, 1, 2, 3), \tag{2.29}$$

and denoting $2a = 2b = -\varkappa$, we finally obtain

$$[\gamma_\mu (\partial_\mu - q_\mu - i\gamma_5 r_\mu) + \varkappa]\psi = 0 \tag{2.30}$$

and

$$[\gamma_\mu (\partial_\mu - p_\mu) + \varkappa]\psi = 0 \tag{2.30'}$$

respectively.

3. *Introduction of new variables*

In Section 2 we have established the correspondence between covariant first and second order differential euqations in the spinor space and the Dirac and Klein-Gordon equations. We have seen that in the general equations (2.18) and (2.30) certain additional differential operators occur containing the explicit derivatives with respect to $z_{\alpha;\beta}$. To solve the general equations we shall have to determine these differential operators by introducing besides the vector variables x_μ another set of four real variables $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ independent of x_μ. In this way it will be possible to consider any function $f(z_{\alpha;\beta}, z_{\dot\alpha;\dot\beta})$ of the four complex (eight real) variables $z_{\alpha;\beta}$ also as a function of the eight real variables x_μ, φ_μ.

$$\begin{aligned} f(z_{\alpha;\beta}, z_{\dot\alpha;\dot\beta}) &= \varphi(x_\mu, \varphi_\mu), \\ x_\mu &= x_\mu (z_{\alpha;\beta}, z_{\dot\alpha;\dot\beta}), \\ \varphi_\mu &= \varphi_\mu (z_{\alpha;\beta}, z_{\dot\alpha;\dot\beta}). \end{aligned} \tag{3.1}$$

In this section we shall show how to introduce the φ_μ and we shall also show that these variables must necessarily possess the character of angles or hyperbolic angles (the notation φ_μ is introduced only for the sake of convenience and the index μ in φ_μ must not be considered as a vector index).

Let us consider the following four domains of space-time:

$$\left.\begin{aligned} &1)\ \text{the inside of the future light-cone, } x_\mu^2 < 0,\ x_0 > 0, \\ &2)\ \text{the inside of the past light-cone, } x_\mu^2 < 0,\ x_0 < 0, \\ &3)\ \text{the part of the outside of the light-cone } x_\mu^2 > 0, \text{ in which } x_0 - x_3 > 0, \\ &4)\ \text{the part of the outside of the light-cone } x_\mu^2 > 0, \text{ in which } x_0 - x_3 < 0. \end{aligned}\right\} \tag{3.2}$$

The sub-division of the outside of the light-cone into two parts is not essential and is introduced here with the purpose to obtain a unified description of the whole of space-time.

For each particular domain we carry out a transformation of $\boldsymbol{c}'$ in such a way as to obtain (we assume ε_μ to be normalized to unity: $\varepsilon_\mu^2 = \pm 1$)

1) For the inside of the future light-cone

$$\varepsilon_\mu^2 = -1, \quad \varepsilon_i = 0, \quad \varepsilon_0 = 1, \quad \varepsilon^{;1\dot 1} = \varepsilon^{;2\dot 2} = -1, \tag{3.3}$$

2) For the inside of the past light-cone

$$\varepsilon_\mu^2 = -1, \quad \varepsilon_i = 0, \quad \varepsilon_0 = -1, \quad \varepsilon^{;1\dot{1}} = \varepsilon^{;2\dot{2}} = 1, \tag{3.4}$$

3) For the outside of the light-cone with $x_0 - x_3 > 0$

$$\varepsilon_\mu^2 = 1, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_0 = 0, \quad \varepsilon_3 = 1, \quad \varepsilon^{;1\dot{1}} = -\varepsilon^{;2\dot{2}} = -1, \tag{3.5}$$

4) For the outside of the light-cone with $x_0 - x_3 < 0$

$$\varepsilon_\mu^2 = 1, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_0 = 0, \quad \varepsilon_3 = -1, \quad \varepsilon^{;1\dot{1}} = -\varepsilon^{;2\dot{2}} = 1. \tag{3.6}$$

In all four cases (3.3—6)

$$\varepsilon^{;1\dot{2}} = \varepsilon^{;2\dot{1}} = 0. \tag{3.7}$$

Introducing (3.7) into (1.14—15) and using the notation without semicolon (cf. (1.11)), we obtain for the connexion between the spinor and the vector space

$$\begin{aligned} x_0 - x_3 &= a_1 |z_1|^2 + a_2 |z_{\dot{1}}|^2, \\ x_0 + x_3 &= a_1 |z_2|^2 + a_2 |z_{\dot{2}}|^2, \\ -x_1 + ix_2 &= a_1 z_1 z_2^* + a_2 z_1^* z_{\dot{2}}, \end{aligned} \tag{3.8}$$

where

$$a_1 = -\varepsilon^{;1\dot{1}}, \quad a_2 = -\varepsilon^{;2\dot{2}}. \tag{3.9}$$

Thus the four domains (3.3—6) are characterized by

$$\begin{aligned} a_1 &= a_2 = 1, \\ a_1 &= a_2 = -1, \\ a_1 &= -a_2 = 1, \\ a_1 &= -a_2 = -1, \end{aligned} \tag{3.10}$$

respectively.

Introducing now the eight real variables

$$z_\alpha = \xi_\alpha + i\eta_\alpha, \quad z_{\dot{\alpha}} = \xi_{\dot{\alpha}} - i\eta_{\dot{\alpha}}, \tag{3.11}$$

we obtain

$$\begin{aligned} x_0 - x_3 &= a_1 (\xi_1^2 + \eta_1^2) + a_2 (\xi_{\dot{1}}^2 + \eta_{\dot{1}}^2), \\ x_0 + x_3 &= a_1 (\xi_2^2 + \eta_2^2) + a_2 (\xi_{\dot{2}}^2 + \eta_{\dot{2}}^2), \\ -x_1 &= a_1 (\xi_1 \xi_2 + \eta_1 \eta_2) + a_2 (\xi_{\dot{1}} \xi_{\dot{2}} + \eta_{\dot{1}} \eta_{\dot{2}}), \\ -x_2 &= a_1 (\xi_1 \eta_2 - \eta_1 \xi_2) + a_2 (\xi_{\dot{1}} \eta_{\dot{2}} - \eta_{\dot{1}} \xi_{\dot{2}}). \end{aligned} \tag{3.12}$$

The geometrical interpretation of these relations is as follows: Consider the three real four-vectors

$$\begin{aligned} \vec{A} &= (\xi_1, \eta_1, \xi_i, \eta_i), \\ \vec{B} &= (\xi_2, \eta_2, \xi_i, \eta_i), \\ \vec{A}' &= (-\eta_1, \xi_1, -\eta_i, \xi_i), \end{aligned} \tag{3.13}$$

in a four-dimensional space with the diagonal metric tensor

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}, \tag{3.14}$$

depending on the domain considered. Relations (3.12) may now be written in the form

$$\begin{aligned} \vec{A}^2 &= x_0 - x_3, \\ \vec{B}^2 &= x_0 + x_3, \\ (\vec{A}\ \vec{B}) &= -x_1, \\ (\vec{A}'\ \vec{B}) &= -x_2, \end{aligned} \tag{3.15}$$

which shows that the lenghts of the vectors $\vec{A}$ and $\vec{B}$ as well as the projections of $\vec{B}$ on $\vec{A}$ and $\vec{A}'$ are determined by the values of x_μ. For fixed x_μ we have still four degrees of freedom left which correspond to rotations or hyperbolic rotations. First we may arbitrarily choose the direction of one of the vectors $\vec{A}$ and $\vec{B}$, say $\vec{A}$. This fixes three of the remaining degrees of freedom, so that the vector $\vec{B}$ of constant lenght may still rotate around the axes $\vec{A}$ and $\vec{A}'$. The remaining one degree of freedom is determined if we fix e. g. the angle between the projection of $\vec{B}$ on the hyperplane $\vec{A}''$, $\vec{A}'''$ and $\vec{A}''$, where $\vec{A}''$ and $\vec{A}'''$ form together with $\vec{A}$ and $\vec{A}'$ an orthogonal basis in the considered space.

As a consequence of this interpretation we introduce the following variables corresponding to the described degrees of freedom

$$\begin{aligned} \xi_1 &= \sqrt{|x_0 - x_3|}\cos a\varphi_2 \cdot \cos\varphi_3, \\ \eta_1 &= \sqrt{|x_0 - x_3|}\cos a\varphi_2 \cdot \sin\varphi_3, \\ \xi_i &= \sqrt{|x_0 - x_3|}\, a^3 \sin a\varphi_2 \cdot \cos\varphi_4, \\ \eta_1 &= \sqrt{|x_0 - x_3|}\, a^3 \sin a\varphi_2 \cdot \sin\varphi_4, \end{aligned} \tag{3.16}$$

where

$$a = \sqrt{a_1 \cdot a_2} = \begin{cases} 1 & \text{if } a_1 = a_2 \\ i & \text{if } a_1 = -a_2, \end{cases} \tag{3.17}$$

and

$$0 \leqslant \varphi_3 \leqslant 2\pi,$$
$$0 \leqslant \varphi_4 \leqslant 2\pi, \tag{3.18}$$
$$0 \leqslant \varphi_2 \leqslant \frac{\pi}{2} + \frac{1}{2}(1 - a^2)\infty.$$

The last of the inequalities (3.18) expresses the fact that for $a = i$, φ_2 is a hyperbolic angle.

To introduce the fourth angle φ_1 we consider the two vectors

$$\vec{A}'' = (-a^2\xi_{\dot{1}}, a^2\eta_{\dot{1}}, \xi_1, -\eta_1),$$
$$\vec{A}''' = (-a^2\eta_{\dot{1}}, -a^2\xi_{\dot{1}}, \eta_1, \xi_1), \tag{3.19}$$

which form together with $\vec{A}$ and $\vec{A}'$ (3.13) an orthogonal set. We easily verify the relation

$$(\vec{A}\,\vec{B})^2 + (\vec{A}'\,\vec{B})^2 + a^2(\vec{A}''\,\vec{B})^2 + a^2(\vec{A}'''\,\vec{B})^2 = \vec{A}^2\,\vec{B}^2 = x_0^2 - x_3^2. \tag{3.20}$$

Together with (3.15) this yields

$$(\vec{A}''\,\vec{B})^2 + (\vec{A}'''\,\vec{B})^2 = -a^2 x_\mu^2 = |x_\mu^2|. \tag{3.21}$$

We introduce, therefore, the fourth angle φ_1 by means of the relations

$$(\vec{A}''\,\vec{B}) = a_2(-\xi_{\dot{1}}\xi_2 + \eta_{\dot{1}}\eta_2 + \xi_1\xi_{\dot{2}} - \eta_1\eta_{\dot{2}}) = \sqrt{|x_\mu^2|}\cos\varphi_1,$$
$$(\vec{A}'''\,\vec{B}) = a_2(-\eta_{\dot{1}}\xi_2 - \xi_{\dot{1}}\eta_2 + \eta_1\xi_{\dot{2}} + \xi_1\eta_{\dot{2}}) = \sqrt{|x_\mu^2|}\sin\varphi_1, \tag{3.22}$$
$$-\pi \leqslant \varphi_1 \leqslant \pi$$

These relations together with the two last relations (3.12) enable us, by virtue of (3.16), to express $\xi_2, \eta_2, \xi_{\dot{2}}, \eta_{\dot{2}}$ by means of x_μ and φ_μ, After a somewhat lenghty calculation we obtain finally in terms of the original complex variables $z_\alpha, z_{\dot{\alpha}}$:

$$z_1 = |x_0 - x_3|^{\frac{1}{2}} e^{i\varphi_3}\cos a\varphi_2,$$
$$z_{\dot{1}} = |x_0 - x_3|^{\frac{1}{2}} e^{-i\varphi_4} a^3 \sin a\varphi_2, \tag{3.23}$$
$$z_2 = |x_0 - x_3|^{-\frac{1}{2}} a_1\{-(x_1 + ix_2) e^{i\varphi_3}\cos a\varphi_2 - e^{i(\varphi_1 - \varphi_4)}\sqrt{|x_\mu^2|}\, a^3 \sin a\varphi_2\},$$
$$z_{\dot{2}} = |x_0 - x_3|^{-\frac{1}{2}} a_1\{-(x_1 - ix_2) e^{-i\varphi_4} a^3 \sin a\varphi_2 + e^{-i(\varphi_1 - \varphi_3)}\sqrt{|x_\mu^2|}\, a^2 \cos a\varphi_2\}.$$

We shall need in the following also the inverse formulae which express x_μ and φ_μ by means of the $z_{\alpha;\beta}, z_{\dot{\alpha};\dot{\beta}}$. The first part of these formulae is already given in equation (3.8). The remaining part may be easily obtained from (3.16) and (3.22):

$$\sqrt{|x_\mu^2|}\, e^{i\varphi_1} = a_2 z_\alpha z^{\dot{\alpha}\star}, \qquad e^{2i\varphi_1} = \frac{z_\alpha z^{\dot{\alpha}\star}}{z_\alpha^\star z^{\dot{\alpha}}} \equiv \frac{z_{\alpha;\beta} z^{\alpha;\beta}}{z_{\dot{\alpha};\dot{\beta}} z^{\dot{\alpha};\dot{\beta}}},$$

$$\cos a\varphi_2 = \frac{|z_1|}{\sqrt{|z_1|^2 + a^2|z_i|^2}}, \quad a^3 \sin a\varphi_2 = \frac{|z_i|}{\sqrt{|z_1|^2 + a^2|z_i|^2}},$$

$$\cos\varphi_3 = \frac{1}{2}(z_1 + z_1^*)\,|z_1|^{-1}, \quad \sin\varphi_3 = \frac{1}{2i}(z_1 - z_1^*)\,|z_1|^{-1},$$

$$\cos\varphi_4 = \frac{1}{2}(z_i + z_i^*)\,|z_i|^{-1}, \quad \sin\varphi_4 = \frac{i}{2}(z_i - z_i^*)\,|z_i|^{-1}. \tag{3.24}$$

We also note the corresponding derivatives $\frac{\partial\varphi_\mu}{\partial z_\varrho}$, $\frac{\partial\varphi_\mu}{\partial z_\varrho^*}$, which will be needed for the determination of the differential operators containing the explicit derivatives $\frac{\partial}{\partial z'_{\alpha;\beta}}$.

$$\frac{\partial\varphi_1}{\partial z_1} = -\frac{i}{2}\frac{z_2^*}{z_\alpha z^{\dot\alpha\star}}, \quad \frac{\partial\varphi_1}{\partial z_2} = \frac{i}{2}\frac{z_1^*}{z_\alpha z^{\dot\alpha\star}}, \quad \frac{\partial\varphi_1}{\partial z_i} = -\frac{i}{2}\frac{z_2^*}{z_\alpha^* z^{\dot\alpha}}, \quad \frac{\partial\varphi_1}{\partial z_{\dot 2}} = \frac{i}{2}\frac{z_1^*}{z_\alpha^* z^{\dot\alpha}},$$

$$\frac{\partial\varphi_2}{\partial z_1} = -\frac{1}{2}\sqrt{\frac{z_1^*}{z_1}}\,\frac{|z_i|}{|z_1|^2 + a^2|z_i|^2}, \quad \frac{\partial\varphi_2}{\partial z_2} = 0,$$

$$\frac{\partial\varphi_2}{\partial z_i} = \frac{1}{2}\sqrt{\frac{z_i^*}{z_i}}\,\frac{|z_1|}{|z_1|^2 + a^2|z_i|^2}, \quad \frac{\partial\varphi_2}{\partial z_{\dot 2}} = 0,$$

$$\frac{\partial\varphi_3}{\partial z_1} = -\frac{i}{2}\frac{1}{z_1}, \quad \frac{\partial\varphi_3}{\partial z_2} = 0, \quad \frac{\partial\varphi_3}{\partial z_i} = 0, \quad \frac{\partial\varphi_3}{\partial z_{\dot 2}} = 0,$$

$$\frac{\partial\varphi_4}{\partial z_1} = 0, \quad \frac{\partial\varphi_4}{\partial z_2} = 0, \quad \frac{\partial\varphi_4}{\partial z_i} = \frac{i}{2}\frac{1}{z_i}, \quad \frac{\partial\varphi_4}{\partial z_{\dot 2}} = 0. \tag{3.25}$$

4. Differential Equations in the Spinor Space in Terms of the Variables x_μ, φ_μ.

Let us now determine finally the form of the equations (2.18), (2.30) and (2.30′) in terms of the variables x_μ, φ_μ. Due to (3.1) and (3.25), we have

$$\frac{\partial}{\partial z_1'} = -\frac{i}{2}\frac{z_2^*}{z_\alpha z^{\dot\alpha\star}}\frac{\partial}{\partial\varphi_1} - \frac{1}{2}\sqrt{\frac{z_1^*}{z_1}}\,\frac{|z_i|}{|z_1|^2 + a^2|z_i|^2}\frac{\partial}{\partial\varphi_2} - \frac{i}{2}\frac{1}{z_1}\frac{\partial}{\partial\varphi_3},$$

$$\frac{\partial}{\partial z_{\dot 2}'} = \frac{i}{2}\frac{z_1^*}{z_\alpha z^{\dot\alpha\star}}\frac{\partial}{\partial\varphi_1},$$

$$\frac{\partial}{\partial z_i'} = -\frac{i}{2}\frac{z_2^*}{z_\alpha^* z^{\dot\alpha}}\frac{\partial}{\partial\varphi_1} + \frac{1}{2}\sqrt{\frac{z_i^*}{z_i}}\,\frac{|z_1|}{|z_1|^2 + a^2|z_i|^2}\frac{\partial}{\partial\varphi_2} + \frac{i}{2}\frac{1}{z_i}\frac{\partial}{\partial\varphi_4},$$

$$\frac{\partial}{\partial z_{\dot 2}'} = \frac{i}{2}\frac{z_1^*}{z_\alpha^* z^{\dot\alpha}}\frac{\partial}{\partial\varphi_1}. \tag{4.1}$$

Introducing (4.1) into (2.19), we get, by virtue of (3.7), (3.9) and (1.11),

$$p^{1\dot 1;} = p_{2\dot 2;} = \frac{i}{2}\frac{x_0 + x_3}{x_\sigma^2}\frac{\partial}{\partial\varphi_1} +$$

$$+\frac{1}{2}\frac{x_1-ix_2}{x_0-x_3}\cdot\frac{e^{i(\varphi_1-\varphi_3-\varphi_4)}}{\sqrt{|x_\sigma^2|}}\left\{\frac{\partial}{\partial\varphi_2}+ia\operatorname{tg}a\varphi_2\frac{\partial}{\partial\varphi_3}-ia\operatorname{ctg}a\varphi_2\frac{\partial}{\partial\varphi_4}\right\}-$$

$$-\frac{i}{2}\frac{1}{x_0-x_3}\left\{\frac{\partial}{\partial\varphi_3}+\frac{\partial}{\partial\varphi_4}\right\},$$

$$p^{1\dot{2}}=-p_{2\dot{1}};=\frac{i}{2}\frac{x_1+ix_2}{x_\sigma^2}\frac{\partial}{\partial\varphi_1}+$$

$$+\frac{1}{2}\frac{e^{i(\varphi_1-\varphi_3-\varphi_4)}}{\sqrt{|x_\sigma^2|}}\left\{\frac{\partial}{\partial\varphi_2}+ia\operatorname{tg}a\varphi_2\frac{\partial}{\partial\varphi_3}-ia\operatorname{ctg}a\varphi_2\frac{\partial}{\partial\varphi_4}\right\},$$

$$p^{2\dot{1}};=-p_{1\dot{2}};=\frac{i}{2}\frac{x_1-ix_2}{x_\sigma^2}\frac{\partial}{\partial\varphi_1},$$

$$p^{2\dot{2}};=p_{1\dot{1}};=\frac{i}{2}\frac{x_0-x_3}{x_\sigma^2}\frac{\partial}{\partial\varphi_1}. \tag{4.2}$$

These formulae determine the form of equations (2.12) or (2.15) in a frame of reference corresponding to the choice of coordinates (3.23). To write equations (2.15) in matrix notation (2.30—30′) we have to calculate from (4.2) the operators q_μ and v_μ. One easily obtains with help of (2.24) and (2.29)

$$q_1=\frac{1}{2}\frac{1}{\sqrt{|x_\sigma^2|}}\cdot\left\{\cos\varphi\frac{\partial}{\partial\varphi_2}-\sin\varphi D_3\right\},$$

$$q_2=\frac{1}{2}\cdot\frac{1}{\sqrt{|x_\sigma^2|}}\cdot\left\{\sin\varphi\frac{\partial}{\partial\varphi_2}+\cos\varphi D_3\right\},$$

$$q_3=\frac{1}{2}\cdot\frac{1}{\sqrt{|x_\sigma^2|}}\cdot\frac{1}{x_0-x_3}\left\{(x_1\cos\varphi+x_2\sin\varphi)\frac{\partial}{\partial\varphi_2}+(-x_1\sin\varphi+x_2\cos\varphi)D_3\right\},$$

$$q_0=q_3,\quad (q_4=iq_0),$$

$$r_1=\frac{x_1}{x_\sigma^2}\frac{\partial}{\partial\varphi_1}+\frac{1}{2}\frac{1}{\sqrt{|x_\sigma^2|}}\left\{\sin\varphi\frac{\partial}{\partial\varphi_2}+\cos\varphi D_3\right\},$$

$$r_2=\frac{x_2}{x_\sigma^2}\frac{\partial}{\partial\varphi_1}+\frac{1}{2}\frac{1}{\sqrt{|x_\sigma^2|}}\left\{-\cos\varphi\frac{\partial}{\partial\varphi_2}+\sin\varphi D_3\right\},$$

$$r_3=\frac{x_3}{x_\sigma^2}\frac{\partial}{\partial\varphi_1}+\frac{1}{2}\frac{1}{\sqrt{|x_\sigma^2|}}\frac{1}{x_0-x_3}\left\{(x_1\sin\varphi-x_2\cos\varphi)\frac{\partial}{\partial\varphi_2}+\right.$$

$$\left.+(x_1\cos\varphi+x_2\sin\varphi)D_3\right\}-\frac{1}{x_0-x_3}D_4,$$

$$r_0=\frac{x_0}{x_\sigma^2}\frac{\partial}{\partial\varphi_1}+\frac{1}{2}\frac{1}{\sqrt{|x_\sigma^2|}}\frac{1}{x_0-x_3}\left\{(x_1\sin\varphi-x_2\cos\varphi)\frac{\partial}{\partial\varphi_2}+\right.$$

$$\left.+(x_1\cos\varphi+x_2\sin\varphi)D_3\right\}-\frac{1}{x_0-x}D_4,$$

where

$$\varphi = \varphi_1 - \varphi_3 - \varphi_4,$$

$$D_3 = a \operatorname{tg} a\varphi_2 \frac{\partial}{\partial \varphi_3} - a \operatorname{ctg} a\varphi_2 \frac{\partial}{\partial \varphi_4},$$

$$D_4 = \frac{1}{2}\left(\frac{\partial}{\partial \varphi_3} + \frac{\partial}{\partial \varphi_4}\right), \tag{4,6}$$

Introducing (4.5) into equations (2.30) and (2.30′) we obtain finally

$$\left[\gamma_\mu\left(\partial_\mu + i\gamma_5 \frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial \varphi_1}\right) + \varkappa + \right.$$

$$+ \frac{1}{2}\frac{e^{i\gamma_5\varphi}}{\sqrt{|x_\sigma^2|}}\left\{-(\gamma_1 - i\gamma_5\gamma_2) + (x_1 - i\gamma_5 x_2)\frac{\gamma_3 + i\gamma_4}{x_3 + ix_4}\right\}\left\{\frac{\partial}{\partial \varphi_2} - i\gamma_5 D_3\right\} +$$

$$\left. + i\gamma_5 \frac{\gamma_3 + i\gamma_4}{x_3 + ix_4} D_4\right] \psi(x_\nu, \varphi_\nu) = 0, \tag{4.7}$$

and

$$\left[\gamma_\mu\left(\partial_\mu - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial \varphi_1}\right) + \varkappa + \right.$$

$$+ \frac{1}{2}\frac{e^{i\varphi}}{\sqrt{|x_\sigma^2|}}\left\{-(-\gamma_1 - i\gamma_2) + (x_1 - ix_2)\frac{\gamma_3 + i\gamma_4}{x_3 + ix_4}\right\}\left\{\frac{\partial}{\partial \varphi_2} + iD_3\right\} -$$

$$\left. - i\frac{\gamma_3 + i\gamma_4}{x_3 + ix_4} D_4\right] \psi(x_\nu, \varphi_\nu) = 0. \tag{4.7′}$$

In the following section we shall consider only a particular case of equation (4.7′) namely the case when the function $\psi(x_\nu, \varphi_\nu)$ does not depend on $\varphi_2, \varphi_3, \varphi_4$. Denoting $\varphi_1 = \alpha$, we then have

$$\left[\gamma_\mu\left(\partial_\mu - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial \alpha}\right) + \varkappa\right] \psi(x_\nu, \alpha) = 0. \tag{4.8′}$$

This equation, as well as equation (4.16), may be solved immediately and are well suited, therefore, for a provisional discussion. The general equations (4.7) and (4.7′) as well as the equation

$$\left[\gamma_\mu\left(\partial_\mu + i\gamma_5 \frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial \alpha}\right) + \varkappa\right] \psi(x_\nu, \alpha) = 0. \tag{4.8}$$

which possess no immediate solutions are now being studied by J. Mozrzymas[2]. It may be noted that equations (4.8) and (4.8′) in contradistinction to (4.7) and (4.7′) are

[2] I would like to express my gratitude to Mr. Mozrzymas for checking the calculations of Sections 3 and 4.

covariant with respect to the full group $\boldsymbol{c} \cdot \boldsymbol{c}'$ due to the invariant definition of α (cf. (3.24)).

The corresponding calculations for the second order differential equation (2.18) are rather involved and, therefore, we shall consider here from the beginning only the particular case when the unknown function does not depend on $\varphi_2, \varphi_3, \varphi_4$. In this case equation (2.18) may be written in the form

$$-\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma} \frac{\partial^2 f}{\partial z_{\varrho\dot\lambda;}\, \partial z^{\varrho\dot\lambda;}} + 2 \frac{\partial z_{\varrho\dot\lambda;}}{\partial z_{\gamma;\beta}} \cdot \frac{\partial \alpha}{\partial z^{\gamma;\beta}} \cdot \frac{\partial^2 f}{\partial z_{\varrho\dot\lambda;}\, \partial \alpha} +$$

$$+ \frac{\partial \alpha}{\partial z_{\gamma;\beta}} \cdot \frac{\partial \alpha}{\partial z^{\gamma;\beta}} \cdot \frac{\partial^2 f}{\partial \alpha^2} + \frac{\partial^2 \alpha}{\partial z_{\gamma;\beta}\, \partial z^{\gamma;\beta}} \cdot \frac{\partial f}{\partial \alpha} = af. \tag{4.9}$$

Now the first row of equations (3.25) may be written

$$\frac{\partial \alpha}{\partial z_{\gamma;\beta}} = \frac{1}{i} \frac{z^{\gamma;\beta}}{z_{\varrho;\sigma} z^{\varrho;\sigma}}. \tag{4.10}$$

We easily obtain therefrom with help of the first of equations (3.24)

$$\frac{\partial^2 \alpha}{\partial z_{\gamma;\beta}\, \partial z^{\gamma;\beta}} = \frac{-i}{z_\sigma z^{\dot\sigma\star}}, \qquad \frac{\partial \alpha}{\partial z_{\gamma;\beta}} \cdot \frac{\partial \alpha}{\partial z^{\gamma;\beta}} = -\frac{1}{2} \frac{1}{z_\sigma z^{\dot\sigma\star}},$$

$$\frac{\partial z_{\varrho\dot\lambda;}}{\partial z_{\gamma;\beta}} \cdot \frac{\partial \alpha}{\partial z^{\gamma;\beta}} = \frac{1}{2i} \frac{z_{\varrho\dot\lambda;}}{z_\sigma z^{\dot\sigma\star}}. \tag{4.11}$$

Introducing (4.11) into (4.9) and dividing by $\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}$ we get, by virtue of (3.24),

$$-\frac{\partial^2 f}{\partial z_{\varrho\dot\lambda;}\, \partial z^{\varrho\dot\lambda;}} + \frac{1}{i} \frac{z_{\varrho\dot\lambda;}}{x_\sigma^2} \cdot \frac{\partial^2 f}{\partial z_{\varrho\dot\lambda;} \partial \alpha} - \frac{1}{2x_\sigma^2} \cdot \frac{\partial^2 f}{\partial \alpha^2} -$$

$$- \frac{i}{x_\sigma^2} \frac{\partial f}{\partial \alpha} = \frac{a}{\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}} f. \tag{4.12}$$

Due to (2.22) and (1.14), we have

$$\partial^{\varrho\dot\lambda;}\, \partial_{\sigma\dot\lambda;} = -\tfrac{1}{2} \Box, \qquad z_{\dot\lambda;}\, \partial^{\varrho\dot\lambda;} = x_\mu \partial_\mu. \tag{4.13}$$

Introducing (4.13) into (4.12), we get

$$\left\{ \Box - \frac{1}{x_\sigma^2} \left[\frac{\partial^2}{\partial \alpha^2} + 2i (x_\mu \partial_\mu + 1) \frac{\partial}{\partial \alpha} \right] \right\} f = \frac{2a}{\varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}} f. \tag{4.14}$$

We see from (4.14) that correspondence with the Klein-Gordon equation may be obtained by putting

$$a = \frac{\varkappa^2}{2} \varepsilon_\mu^2 z_\sigma^\star z^{\dot\sigma}. \tag{4.15}$$

Representing the operator on the left-hand side of (4.14) as the square of a first order differential operator and choosing a according to (4.15), we may write finally equation (4.14) in the form

$$\left(\partial_\mu - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial\alpha}\right)^2 f(x_\nu, \alpha) = \varkappa^2 f(x_\nu, \alpha). \tag{4.16}$$

5. *Solutions of the Equations*

We shall consider here, for simplicity, only the solutions of equations (4.8′) and (4.16), which may give us already some insight into the possibilities offered by equations of the general type (4.7) or (4.7′). Let us start with equation (4.8′).

Separation of α

$$\psi(x_\nu, \alpha) = e^{im\alpha}\varphi(x_\nu), \tag{5.1}$$

brings (4.8) into the form ($\lambda = x_\sigma^2$)

$$\left[\gamma_\mu\left(\partial_\mu + m\frac{x_\mu}{\lambda}\right) + \varkappa\right]\varphi(x_\nu) = 0. \tag{5.2}$$

We try to solve equation (5.2) by the help of the „Ansatz"

$$\varphi(x_\nu) = f(x_\nu)\,\chi(x_\nu), \tag{5.3}$$

where $\chi(x_\nu)$ is the solution of the Dirac equation

$$(\gamma_\mu\partial_\mu + \varkappa)\,\chi(x_\nu) = 0. \tag{5.4}$$

Introducing (5.3) into (5.2) we obtain, due to (5.4),

$$\frac{\partial f}{\partial x_\mu} + m\frac{x_\mu}{\lambda}f = 0. \tag{5.5}$$

Equation (5.5) is solved by the function

$$f(\lambda) = |\lambda|^{-\frac{m}{2}} \tag{5,6}$$

so that[3]

$$\varphi(x_\nu) = |\lambda|^{-\frac{m}{2}}\chi(x_\nu). \tag{5.7}$$

A general solution of (4.8′) may be written as a superposition of the partial solutions (5.1)

$$\psi(x_\nu, \alpha) = \sum_{m=-\infty}^{+\infty}\left(\frac{e^{i\alpha}}{\sqrt{|x_\sigma^2|}}\right)^m \chi_m(x_\nu), \tag{5,8}$$

[3] I am indebted to Mr. A. Pawlikowski for finding solution (5.7) and for many stimulating discussions.

where the $\chi_m(x_\nu)$ $(m = 0, \pm 1, \ldots)$ are arbitrary solutions of the Dirac equation (5.4). We may write this solution also in the form

$$\psi(x_\nu, \alpha) = F(x_\nu, z), \tag{5.9}$$

where

$$z = \frac{e^{i\alpha}}{\sqrt{|x_\sigma^2|}}, \tag{5.10}$$

and $F(x_\nu, z)$ is an arbitrary function of the complex variable z satisfying the Dirac equation (5.4) with respect to the variables x_ν occuring explicitly in $F(x_\nu, z)$ (not by means of z!).

It may be easily verified that (5.9) is also a solution of the second order equation (4.16). Indeed,

$$\left(\partial_\mu - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial\alpha}\right)F(x_\nu, z) = \partial_\mu^{ex} F(x_\nu, z) + \\ + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x_\mu} - i\frac{x_\mu}{x_\sigma^2}\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial\alpha}, \tag{5.11}$$

where ∂_μ^{ex} denotes the derivatives with respect to the x_μ occuring explicitly in F. But

$$\frac{\partial z}{\partial x_\mu} = -\frac{x_\mu}{\lambda}z, \qquad \frac{\partial z}{\partial\alpha} = iz, \tag{5.12}$$

and the last two terms in (5.11) cancel. By a second differentiation we get, therefore,

$$\left(\partial_\mu - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial\alpha}\right)^2 F(x_\nu, z) = \partial_\mu^{ex}\partial_\mu^{ex} F(x_\nu, z) = \varkappa^2 F(x_\nu, z), \tag{5.13}$$

the last equality being due to the assumed property

$$(\gamma_\mu \partial_\mu^{ex} + \varkappa) F(x_\mu, z) = 0, \tag{5.14}$$

of the function $F(x_\nu, z)$.

A particular solution is selected by determining the initial values in the space of the variables x_μ and α. However, without specifying the initial conditions, we see that a physically admissible solution must necessarily be of the form

$$F(x_\nu, z) = \frac{f(x_\nu, z)}{\prod_i\left(1 - \frac{z}{z_i}\right)}, \tag{5.15}$$

where $f(x_\nu, z)$ is a regular function of z satisfying equation (5.14). If we demand also regularity on the light-cone ($|z| = \infty$), $f(x_\nu, z)$ must behave properly for $|z| \to \infty$, namely in such a way that the function $F(x_\nu, z)$ vanishes or approaches a constant.

For $z \to 0$, i. e. for $\lambda \to \infty$,

$$F(x_\nu, z) \to f(x_\nu, 0). \tag{5.16}$$

Of course (5.15) is a solution of (4.8′) or (4.16) only outside the poles z_i.

We may easily show that a function of the type (5.15) is integrable in the same sense as the corresponding solution of the Dirac equation. Carrying out the extra integration over α we get namely

$$\bar{F}(x_\nu) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} F(x_\nu, z)\, d\alpha = \frac{1}{2\pi i} \oint \frac{F(x_\nu, z)}{z}\, dz, \tag{5.17}$$

where the contour of integration is a circle around the point $z = 0$ ($\lambda = \infty$) with a radius

$$|z| = \frac{1}{\sqrt{|x_\sigma^2|}}. \tag{5.18}$$

Thus, for a fixed x_σ^2, (5.17) is just the sum of the residua corresponding to the poles enclosed by this circle (we assume for simplicity only first order poles)

$$\bar{F}(x_\nu) = \frac{1}{2\pi i} \oint \frac{f(x_\nu, z)\, dz}{z \prod_i \left(1 - \frac{z}{z_i}\right)}$$

$$= f(x_\nu, 0) - \sum_{k=1}^{n(\lambda)} \frac{f(x_\nu, z_k)}{\prod_i{}' \left(1 - \frac{z_k}{z_i}\right)}. \tag{5.19}$$

Each term in the sum is a solution of the Dirac equation. The number of terms $1 + n(\lambda)$ is a step-function of $|\lambda|$ which remains constant if $|\lambda|$ is contained between any two successive values $|\lambda_i|$ and $|\lambda_k|$ corresponding to two radii of two successive poles z_i and z_k and which increases by one if $|\lambda|$ passes decreasing one of these values. Thus for

$$\frac{1}{|z_k|^2} < |\lambda| < \frac{1}{|z_i|^2}, \qquad (|z_k| > |z_i|), \tag{5.20}$$

where z_i and z_k denote any two successive poles, the function $\bar{F}(x_\nu)$ is a solution of the Dirac equation. On the hypersurfaces

$$|\lambda| = \frac{1}{|z_i|^2}, \quad (i = 1, 2, \ldots), \tag{5.21}$$

$\bar{F}(x_\nu)$ is discontinuous. The function $F(x_\nu, z)$ possesses, therefore, the same properties with respect to integrability as the solutions $f(x_\nu, 0)$, $f(x_\nu, z_k)$ of the Dirac equation.

It is seen that the process of averaging over the variable α (5.17) provides another method of transition to the solutions of the Dirac equation as compared with the method of Section 2 where transition to the Dirac equation was obtained by simply neglecting the dependence on the variables φ_μ. If there are no poles in (5.15) we have simply

$$\overline{F}(x_\nu) = F(x_\nu, 0) \tag{5.22}$$

In any case, the dependence of $F(x_\nu, z)$ on λ by the intermediary of z possesses the character of a form-factor describing the structure of the particle. The transition to a point particle, or rather to a plane wave, is obtained by the averaging process (5.17). It is, therefore, tempting to consider the general equations (4.7) or (4.7′) as connected in some way with the internal structure of fundamental particles.

In this connexion it may be interesting to consider the Lagrangean formalizm associated with equation (4.8′) and to derive the generalized five-current.

Equation (4.8′)

$$\left[\gamma_\mu\left(\vec{\partial}_\mu - i\frac{x_\mu}{x_\sigma^2}\frac{\vec{\partial}}{\partial\alpha}\right) + \varkappa\right]\psi(x_\nu, \alpha) = 0, \tag{5.23}$$

and the complex conjugate equation

$$\overline{\psi}(x_\nu, -\alpha)\left[\gamma_\mu\left(\overleftarrow{\partial}_\mu - i\frac{x_\mu}{x_\sigma^2}\frac{\overleftarrow{\partial}}{\partial\alpha}\right) - \varkappa\right] = 0, \tag{5.24}$$

may be derived from the Lagrangean

$$\mathcal{L}(x_\nu, \alpha) = \overline{\psi}(x_\nu, -\alpha)\left[\gamma_\mu\left(\partial_\mu - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial\alpha}\right) + \varkappa\right]\psi(x_\nu, \alpha). \tag{5.25}$$

Here $\overline{\psi}(x_\nu, \alpha) = \psi^*(x_\nu, \alpha)\,\gamma_4$ and the arrows in (5.23—24) indicate the direction in which the differentiations have to be carried out. One easily verifies the relation

$$\mathcal{L}^+(x_\nu, \alpha) = \mathcal{L}(x_\nu, -\alpha) - \partial_\mu j_\mu(x_\nu, \alpha) - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial\alpha} j_\mu(x_\nu, -\alpha), \tag{5.26}$$

where

$$j_\mu(x_\nu, \alpha) = \overline{\psi}(x_\nu, -\alpha)\,\gamma_\mu\,\psi(x_\nu, \alpha), \tag{5.27}$$

and $\mathcal{L}^+$ is the hermitean conjugate to $\mathcal{L}$

Due to (5.26) the action functional

$$W = \int dx \cdot \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\alpha\,\mathcal{L}(x_\nu, \alpha) \tag{5.28}$$

is real

$$W^+ = W.$$

By virtue of (5.23—24) we have on the extremals of W the conservation law

$$\partial_\mu j_\mu(x_\nu, \alpha) - i \frac{x_\mu}{x_\sigma^2} \frac{\partial}{\partial \alpha} j_\mu(x_\nu, \alpha) = 0. \tag{5.30}$$

By averaging (5.30) over α we get the ordinary conservation law

$$\partial_\mu \bar{j}_\mu(x_\nu) = 0 \tag{5.31}$$

for the quantity

$$\bar{j}_\mu(x_\nu) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} j_\mu(x_\nu, \alpha)\, d\alpha. \tag{5.32}$$

Thus $j_\mu(x_\nu, \alpha)$ may be considered to be the generalized current corresponding to the ordinary Dirac current $\bar{j}_\mu(x_\nu)$. This generalized current exhibits, due to (5.15), the structure of the elementary particle. This structure gets lost by the averaging process (5.32) giving place to a uniform distribution corresponding to a plane wave.

We hope to give a more detailed discussion of these questions at another place.

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ISOMERIC TRANSITION IN Hg^{199}

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The spectrum of internal conversion electrons for the isomeric transition in Hg^{199} from the $i_{13/2}$ to the $f_{5/2}$ levels was investigated. The energy of the transition was found to be $E = 371.1 \pm 3.5$ KeV and the ratios of the internal conversion coefficients $K:L:(M+N) = 1:(0.57 \pm 0.09):(0.12 \pm 0.07)$. It was found from the $K:L$ ratio that the percent admixture of type *E5* transitions to the basic *M4* did not exceed 11%.

Introduction

The isotope Hg^{199} has a 44 minute isomeric state (Strominger et al. 1958) which decays into the ground state by two succesive γ transitions of 370 KeV and 158 KeV, respectively. The 158 KeV line is also observed in the β^- decay of Au^{199} and in electron capture by the Tl^{199} nucleus; it has been investigated frequently and has been identified as an E2 transition (e. g. Bäckström et al. 1958). The Hg^{199} isomer cannot arise from Au^{199} because of energetic considerations; it also cannot arise from Tl^{199} owing to the large difference in spins. It is obtained in the reactions (n, n), $(n, 2n)$, and (γ, n), and thus the reaction products cannot be separated by radiochemical methods. This makes it difficult to obtain sources with specific activity suitable for investigation in a magnetic spectrometer. Consequently, the transition with an energy of 370 KeV is known less accurately. In particular, despite the known fact that the basic transition is of the fourth order magnetic type *M4* (Goldhaber and Hill 1952), the contribution of the transition of the fifth order electric type E5 has not been established. Measurement of the angular correlations points to 0.8% (Pound and Wertheim 1956) and $(30 \pm 20)\%$ (Bolotin and Wilkinson 1955) of *E5*. The size of the *E5* contribution can be estimated on the basis of the absolute coefficient of internal conversion in the *K* shell. A comparison of the value obtained by Bolotin and Wilkinson (1955) with the data from Sliv's and Band's tables (1956) yields a 90% admixture of *E5*.

Another method for estimating the contribution of *E5* to the transition consists in measuring the ratio of the coefficients of internal conversion in the *K* and *L* shells. The internal conversion electron spectrum was investigated by Hole (1948 and 1949);

however, owing to the considerable thickness of the source, he was unable to estimate the contributions of electrons from the M and N levels to the L line, and therefore did not determine the ratio K/L.

The aim of this work was to investigate the spectrum of the internal conversion electrons for the 370 KeV transition under conditions that allowed measurement of the ratio K/L and to estimate the $E5$ contribution by comparison with the data of Sliv and Band (1956 and 1958) and Rose (1958).

Equipment and Measurements

To measure the spectrum of the internal conversion electrons, the authors employed a magnetic spectrometer with a thick lens similar to that described by Siegbahn (1946). The detector was a G-M counter of the BAT-10[1)] type with a mica window 1.3 mg/cm² thick. The absorption of the electrons in the window was negligibly small. The counter background did not exceed 3 cpm. For a thin source diameter and counter diaphragm diameter of 5 mm the resolving power of the spectrometer was 3.3%.

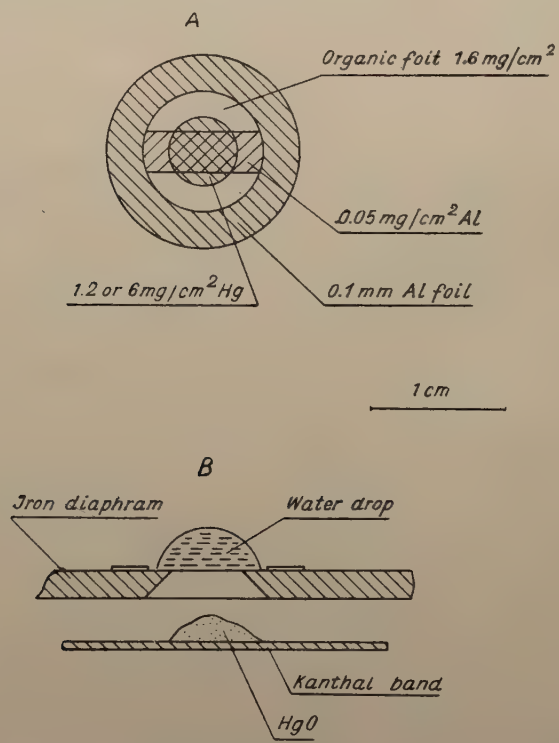

Fig. 1. Source with support (A) and method of preparing source (B).

[1] Produced by the 3rd Division of the Institute for Nuclear Research.

The Hg^{199} source was prepared in the following way: A layer of metallic mercury was applied to an organic foil 1.6 mg/cm² thick which was glued with shellac to a thin aluminium ring. This layer was produced by condensing on the water-cooled foil the mercury vapour obtained from the dissociation of HgO heated to a temperature of about 400°C (Fig. 1)[2]. Next, the mercury was covered with a layer of collodium of thickness about 0.01 mg/cm². To avoid charging the source, the side opposite the mercury was sputtered with 0,05 mg/cm² of aluminium. The source prepared in this way was irradiated through a cadmium filter for two hours in the reactor of the Institute for Nuclear Research. In order to reduce the background from the activated source support the measurement was begun 30 minutes after completion of the irradiation. The spectrum was measured for sources 6 and 1.2 mg/cm² thick; however, it was found that the thicker source could not be used to determine the K/L ratio.

Results

After each irradiation, the activity of the source was sufficient for two hours of measurements. This was the time needed for one measurement of the spectrum within the limits of $H\varrho = 2600$ gauss-cm and $H\varrho = 1550$ gauss-cm.

The results of one measurement are shown in Fig. 2. In order to obtain the spectrum of internal conversion electrons, the decay of activity of the source should be

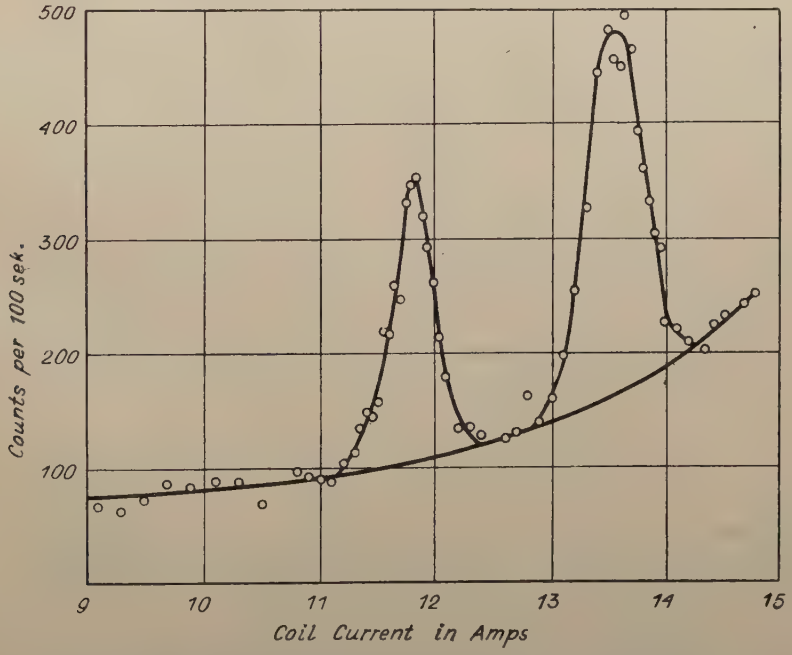

Fig. 2. Results of one measurement for a source 1.2 mg/cm² thick.

[2] The operation was performed in the air under normal preasure.

taken into account after subtracting the background. To decrease the relative statistical error, the results of two measurements were added together. The momentum distribution of the spectrum obtained in this way is shown in Fig. 3.

Comparison of the half-width and shape of the K and $L+M+N$ lines indicates that the contribution of the $M+N$ line is not negligibly small. In order to separate

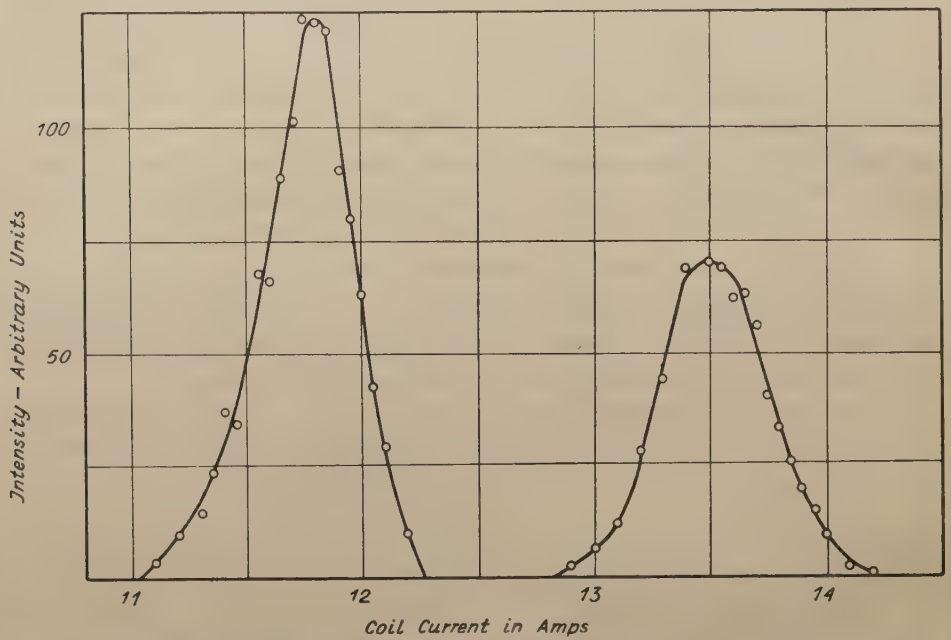

Fig. 3. Momentum distribution of the K and $L + M + N$ lines for a source 1.2 mg/cm² thick.

the $L + M + N$ line into its components L and $M+N$, the shape of the lines L_I, L_{II}, L_{III}, and $M + N$ was assumed to be identical to that of the K line. The ratios of the intensities L_I: L_{II}: L_{III} were taken from the tables of Sliv and Band (1958). Then, by means of Pearson's χ^2 test, the most probable intensities of the L and $M + N$ lines were found. In this way, the ratios

$$K/L = 1.74 \text{ and } [M + N]/K = 0.13$$

were obtained.

A comparison of the experimental points with the best *fit* obtained in this way is seen in Fig. 5. It follows from the figure that the authors' assumptions as to the identity in shape of the L, $M+N$, and K lines was not entirely correct. This is understandable, since the difference in the binding energy for the K and L shells is rather considerable, and consequently the influence of the source thickness is greater for the K line. Although the measured K/L ratio should not be very sensitive to the choice of shape for the line, the authors repeated the separation of the $L+M+N$

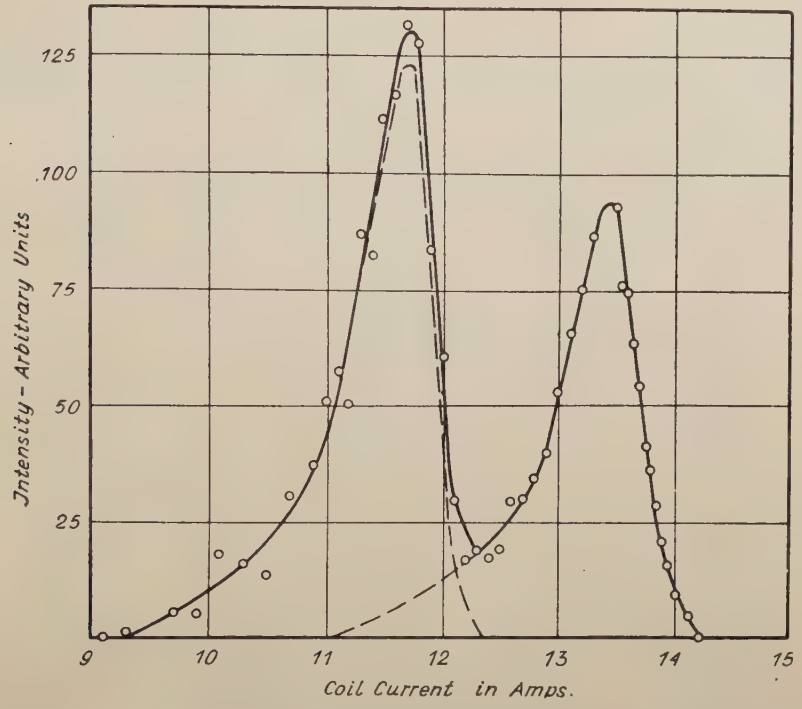

Fig. 4. Momentum distribution of the K and $L + M + N$ lines for the 6 mg/cm² source. The diffuseness of the lines caused by the thickness of the source did not permit an estimate of the $M + N/L$ ratio.

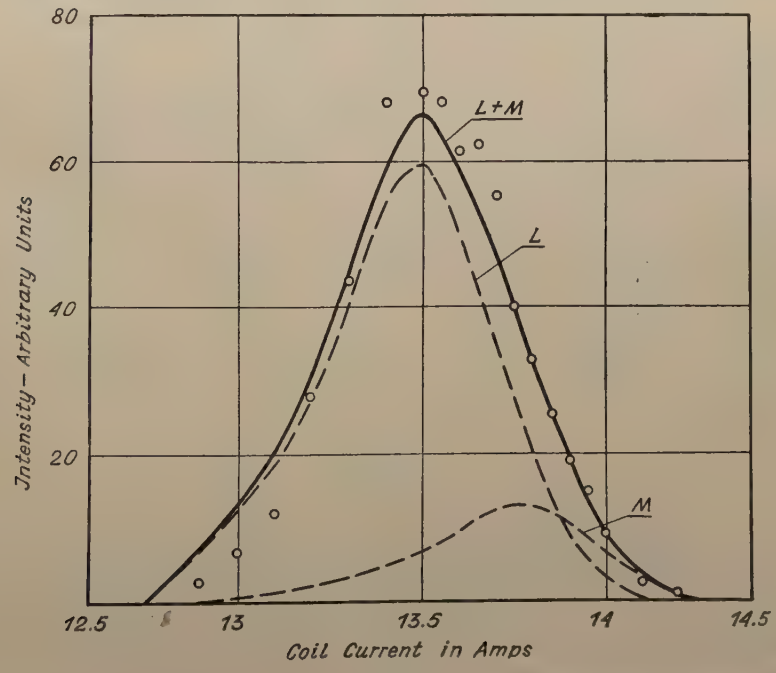

Fig. 5. Separation of the $L + M + N$ line into the L and $M + N$ components for a source 1.2 mg/cm² thick. The shape of the K line was taken as a standard.

line for the assumption that the lines L_I, L_{II}, L_{III}, and $M+N$ are more symmetric than the K line. The magnitude of the assymmetry was determined from the calculated position of the L_I line and the low energy limit of the $L+M+N$ line. Using the same χ^2 test as before resulted in

$$K/L = 1.75 \quad (M+N)/K = 0.11$$

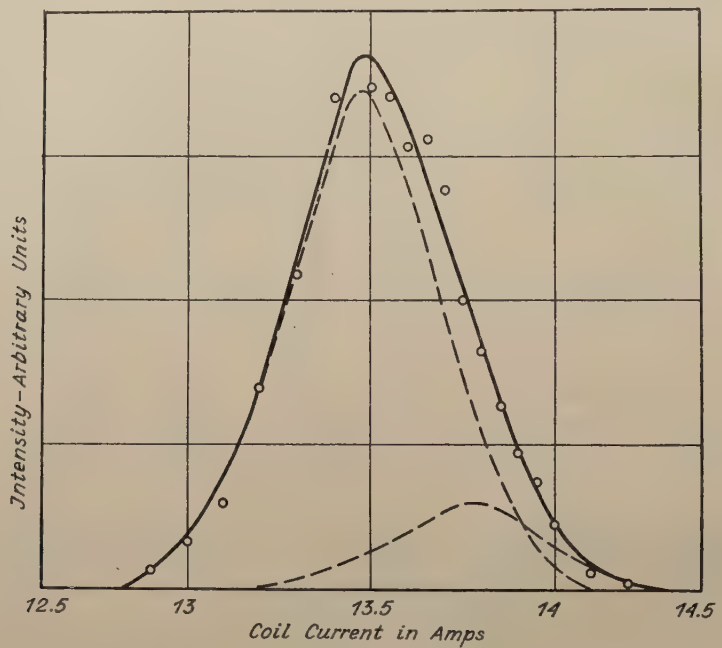

Fig. 6. Separation of the $L+M+N$ line into the L and $M+N$ components for a source 1.2 mg/cm². The shape of a partially symmetrized K line was taken as a standard.

The fit for the $L+M+N$ curve for this case is shown in Fig. 6. Moreover, measurement of the spectrum of the internal conversion electrons admits of an accurate measurement of the energy of the transition. The value

$$E = 371.1 \pm 3.5 \text{ KeV}$$

is obtained.

Discussion

The final results of the measurements are as follows: $K{:}L{:}\,(M{+}N) = 1{:}(0.57 \pm \pm 0.09){:}(0.12 \pm 0.07)$. According to the tables of Sliv and Band (1956 and 1958), these should be $K{:}L = 1{:}0.57$, and according to the tables of Rose (1958) — $K{:}L{:}M = 1{:}0.59{:}0.28$.

As may be seen, the measured K:L ratio is in good agreement with the theoretical results calculated for the $M4$ transition under the assumption that the nucleus has finite dimensions and accounting for the screening of the field of the nucleus by the atomic electrons. As may be seen in Fig. 7, the maximum admixture of the $E5$ multipole does not exceed 11%. The authors' result is therefore in agreement with that obtained by Pound and Wertheim (1956) and close to the data of Bolotin and

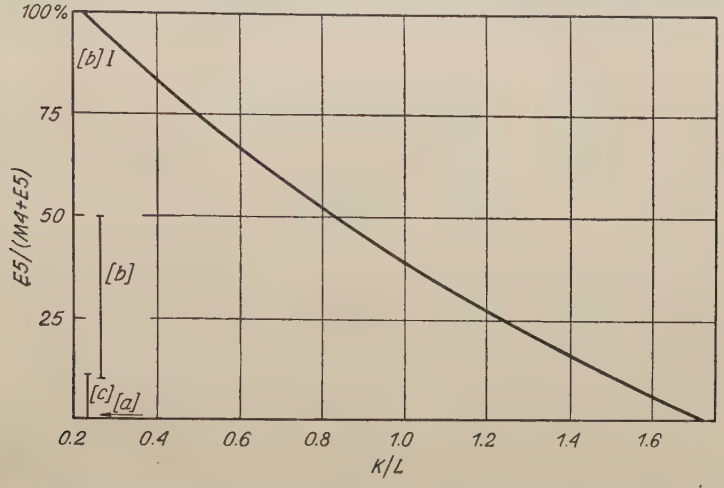

Fig. 7. Comparison of the degree of mixture of the $M4$ and $E5$ multipoles measured by: (a) Pound and Wertheim (1956), (b) Bolotin and Wilkinson (1955) and (c) the present authors.

Wilkinson (1955) obtained by the method of angular correlation, and in complete disagreement with that part of the work of these authors in which they measured the absolute coefficient of internal conversion.

The measured K:$(M+N)$ ratio is considerably greater than the K:M ratio predicted by Rose. It should be remembered, however, that Rose's results for the M shell were calculated without accounting for the screening effect. Inclusion of this effect might diminish the coefficient of internal conversion by as much as 50% (Listengarten 1958).

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Q-ZAHL-VERTAUSCHUNGSREGELN UND PHYSIKALISCHER INHALT EINER EINFACHEN FELDTHEORIE

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It is shown that there exist q-number-commutationrelations with the following properties: (1) They are relativistically invariant and microscopically causal. (2) The physical content of the isolated field is as usual; especially we have particles as eigenstates of the field. (3) The field interacts with other fields in another way, unconventionally. These facts are proved for a real scalar field with rest-mass zero.

1. Einführung

Eine Reihe von Untersuchungen [1—4] über die Frage der Messbarkeit der Feldgrößen führte uns zu folgendem Resultat: Wenn man die Beschränkungen der Messbarkeit von Feldern, welche durch die quantenhafte Natur aller realen Probekörper hervorgerufen werden, sachgemäß in die Feldtheorie einbaut, dann sind die differentiellen Feldgleichungen i. a. verschieden von den konventionellen und die Kommutatoren zwischen den Feldgrößen sind selbst Operatoren.

Während die Feldgleichungen in [4] explizit (für einfache Modelle) angegeben wurden, können die Vertauschungsregeln naturgemäß nicht vollständig aus allgemeineren Prinzipien abgeleitet werden. Wir wollen deshalb unten über die Vertauschungsregeln einfache Annahmen machen und sehen, was daraus folgt. Da meines Wissens q-Zahl Vertauschungsregeln bisher für Feldgrößen noch nie untersucht wurden, ist es wichtig zu bemerken, daß es nach [4] auf jeden Fall ein spezielles Feld gibt (isoliertes elektromagnetisches Feld), bei dem die modifizierten differentiellen Feldgleichungen wieder die konventionelle Form haben. Mann kann also unser Programm in zwei Schritten weiterführen: a) Welchen Inhalt erhält eine Feldtheorie, wenn die konventionellen Feldgleichungen und q-Zahl-Vertauschungsregeln zu Grunde gelegt werden. b) Was geschieht physikalisch, wenn man darüber hinaus die Feldgleichungen entsprechend [4] abändert.

Wir befassen uns hier nur mit der Teilfrage a). Die Diskussion einer solchen Fragestellung scheint auch deshalb lohnend zu sein, weil ja nach Elimination der unphysikalischen Zustände, die in einfachen Modell-Feldtheorien auf jeden Fall auf-

treten, mittels der Methode von Bogoljubow [5] die resultierenden Feldgrößen ebenfalls q-Zahl-Vertauschungsregeln genügen [6].

Verständlicherweise interessiert es uns besonders, ob man mit q-Zahl-Vertauschungsregeln das Divergenzproblem bei wechselwirkenden Feldern angreifen kann. In diesem Zusammenhang dürfte folgende Tatsache von Bedeutung sein: Es ist möglich, die q-Zahl-Vertauschungsregeln zwischen den Feldgrößen einer sehr einfachen Modell-Theorie so zu wählen, daß das isolierte Feld dieselben physikalischen Eigenschaften hat wie das in konventioneller Weise quantisierte isolierte Feld. Erst bei Einschalten einer geeigneten Wechselwirkung unterscheidet sich die Feldtheorie mit q-Zahl-Vertauschungsregeln von der konventionellen. Den ersten Teil der eben formulierten Behauptung beweisen wir im nächsten §, den zweiten im § 3. Im Anhang setzen wir uns mit den von Lehmann bewiesenen Sätzen [10] über die Kommutator-Funktionen auseinander.

Es sei gleich hier darauf hingewiesen, daß der Sinn dieser Arbeit nicht darin liegt, das Divergenzproblem selbst zu lösen. Verf. scheint aber die Erkenntnis wichtig zu sein, daß durch Übergang zu q-Zahl-Vertauschungsreqeln die Wechselwirkung verändert werden kann, unter Beibehaltung der Eigenschaften des isolierten Feldes.

2. *Isoliertes Feld*

Wir legen unserer Betrachtung wieder das denkbar einfachste, reelle Modell-Feld $\Phi(x)$ mit der Feldgleichung

$$\Box \Phi(x) = 0 \tag{1}$$

oder der Lagrange-Dichte

$$L = \tfrac{1}{2}\, \partial_\mu \Phi \cdot \partial^\mu \Phi \tag{2}*$$

zu Grunde.

Als Vertauschungsregel setzen wir fest:

$$[\Phi(x), \Phi(x')] = iD(x - x') \cdot \Gamma. \tag{3}**$$

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* Genau genommen sind (1) und (2) nur äquivalent, wenn für alle Variationen $\delta\Phi$ von Φ die Relation $[\delta\Phi(x), \Phi(y)] = 0$ gilt oder wenn $[\delta\Phi(x), \Phi(y)]$ gewissen einschränkenden Bedingungen genügt.

** D ist die konventionelle D-Funktion mit den Eigenschaften:

$$\left.\begin{aligned} \partial^\mu D(x) &= 0 \\ \partial^\mu D(x) &= +\,\delta^\mu(x) \end{aligned}\right\} \text{ für raumartige } x$$

δ^μ ist definiert durch $\int\limits_\sigma f(x')\delta^\mu(x - x')\, d\sigma^\mu(x') = f(x)$ für $x \varepsilon \sigma$; σ = raumartige Hyperfläche. Deshalb ist (3) auch gleichbedeutend mit:

$$\left.\begin{aligned} [\Phi(x), \Phi(x')] &= 0 \\ [\Phi(x), \Phi^{\mu\prime}(x')] &= -\,i\delta^\mu(x - x')\,.\,\Gamma \end{aligned}\right\} \text{ für raumartige } x\text{—}x.' \tag{4}$$

Hier und im folgenden ist $\Phi^\mu(x) = \partial^\mu \Phi(x)$.

Γ soll eine skalare, von x und x' unabhängige, zunächst ganz beliebige q-Zahl sein. Wir werden Γ gewissen Bedingungen unterwerfen müssen, damit die durch (1) und (3) charakterisierte Theorie die oben erläuterten Eigenschaften hat.

Da Γ als skalare, konstante Größe vorausgesetzt wird, sind (3) und (1) natürlich verträglich und die Theorie ist relativistisch invariant und mikroskopisch kausal.

Es sei auch darauf hingewiesen, daß unser Ansatz (3) oder (4) mit der aus dem Schwingerschen Wirkungsprinzip für obiges L, (2), fließenden Bedingung an die Kommutatoren (vgl. etwa [7], S. 22)

$$\delta\Phi(x) = \frac{i}{2}\left[\Phi(x), \int_\sigma \left(\frac{\partial L(y)}{\partial \Phi^\mu(y)} \cdot \delta\Phi(y) + \delta\Phi(y) \cdot \frac{\partial L(y)}{\partial \Phi^\mu(y)}\right) d\sigma_\mu(y)\right] \quad (5)^{***}$$

verträglich ist.

Denn nach Einsetzen von (2) und (4) sowie unter der Voraussetzung $[\Phi(x), \delta\Phi(y)] = 0$ für $x - y$ raumartig (6) erhält man für (5):

$$\delta\Phi(x) = \tfrac{1}{2}(\Gamma\,\delta\Phi(x) + \delta\Phi(x)\,\Gamma).$$

Das ist natürlich erfüllt für $\Gamma = 1$, aber auch dann, wenn Γ die Form

$$\Gamma = 1 + F, \text{ mit } \delta\Phi(x)\,F + F\,\delta\Phi(x) = 0 \quad (7)$$

hat. Die Voraussetzung (7) werden wir in Zukunft in der etwas veränderten Form:

$$\Gamma = 1 + F, \; \varphi(x)\,F + F\,\varphi(x) = 0 \quad (7a)$$

noch mehrfach zu benutzen haben. Nun zeigen wir, daß unser durch (1) und (3) oder (1) und (4) charakterisiertes Feld Teilchen enthält, die sich in nichts von den Teilchen des Feldes mit $\Gamma = 1$ unterscheiden, wenn nur stets (7a) gilt. Der Beweis hierfür läßt sich wohl am einfachsten führen, indem man zeigt, daß die Größen P_μ* und $M_{\mu\nu}$** in unserem Modell mit Γ gemäß (7a) dieselben Vertauschungsregeln mit den Feldgrößen Φ erfüllen wie im Falle $\Gamma = 1$. Denn aus den Vertauschungsregeln $[P_\mu, \Phi]$ und $[M_{\mu\nu}, \Phi]$ folgen ja direkt die physikalisch vernünftigen Eigenwerte von P_μ und $M_{\mu\nu}$, die die bekannte Teilchenzahl-Interpretation zulassen (vgl. etwa [8], bes. S. 27 ff.).

Wir beachten zuerst, daß (vgl. etwa [7], S. 19—20):

$$P^\mu = \int_\sigma T^{\mu\nu}\, d\sigma_\nu; \quad (8)$$

$$M^{\mu\nu} = \int_\sigma (T^{\varrho\mu} x^\nu - T^{\varrho\nu} x^\mu)\, d\sigma_\varrho, \text{ mit} \quad (9)$$

$$T^{\mu\nu} = \tfrac{1}{2}(\Phi^\mu \Phi^\nu + \Phi^\nu \Phi^\mu - \delta^{\mu\nu} \Phi^\lambda \Phi_\lambda) \text{ gilt,} \quad (10)$$

*** Die Integration erstreckt sich über eine beliebige raumartige Hyperfläche σ, mit $x \varepsilon \sigma$.

* P_μ = Vierervektor von Gesamt-Energie und Gesamt-Impuls des Feldes.

** $M_{\mu\nu}$ = Relativistisch verallgemeinerter Gesamt-Drehimpuls des Feldes.

ganz unabhängig von allen Vertauschungsregeln, also sowohl für die konventionelle wie für unsere Theorie.*

Die Integrationen in (9) und (10) sind über beliebige raumartige Hyperflächen σ zu erstrecken. Diese Willkür in der Lage von σ nutzt man bei der Berechnung von $[P^\mu, \varphi(x)]$ zweckmäßig so aus, daß $x \varepsilon \sigma$ wird. Dann kann man nämlich sofort (vgl. (4)) die Relationen

$$[\Phi(x), \Phi^\mu(x')] = -i\delta^\mu(x-x')(1+F)$$

und $\int\limits_{x\varepsilon\sigma} \delta_\mu(x-x')\, G(x')\, d\sigma_\mu(x') = G(x)$ verwenden.

So ergibt sich:

$$[P^\mu, \Phi(x)] = i\Phi^\mu(x) + \frac{i}{2}\int\limits_{x\varepsilon\sigma} \{\delta^\mu(x-x')\, F\Phi^\nu(x') +$$

$$+ \Phi^\mu(x')\,\delta^\nu(x-x')\,F + \delta^\nu(x-x')\,F\Phi^\mu(x') + \Phi^\nu(x')\,\delta^\mu(x-x')\,F$$

$$- \delta^{\mu\nu}(\delta^\lambda(x-x')\,F\Phi_\lambda(x') + \Phi^\lambda(x')\,\delta_\lambda(x-x')\,F)\}\, d\sigma_\nu(x').$$

Hier sieht man sofort, daß wegen (7a):

$$F\Phi(x) + \Phi(x)\,F = 0, \text{ also auch}$$

$$F\Phi^\mu(x') + \Phi^\mu(x')\,F = 0 \text{ folgt:}$$

$$[P^\mu, \Phi(x)] = i\Phi^\mu(x) \tag{11}$$

Das ist die konventionelle Relation, aus der die bekannten Eigenwerte von P^μ folgen; hieran wird also durch die Anwesenheit des Operators $F \neq 0$ nichts geändert. Insbesondere folgt aus (11), daß die Fourier-Komponenten von $\Phi(x)$ in üblicher Weise als Erzeugungs- bzw. Vernichtungsoperatoren für Teilchen bestimmten Impulses gedeutet werden können, obgleich die Vertauschungsrelationen zwischen diesen Fourierkomponenten wiederum die Größe Γ enthalten. Diese Bemerkung ist entscheidend für die im § 3 folgende Diskussion gewisser Vakuum-Erwartungswerte. Ganz analog kann man zeigen, daß unter der gleichen Voraussetzung über F gilt:

$$[M^{\mu\nu}, \Phi(x)] = i(\Phi^\mu(x)\cdot x^\nu - \Phi^\nu(x)\,x^\mu). \tag{12}$$

Die zunächst auftretenden Zusatzglieder haben nämlich die Form $F\Phi^\mu x^\nu + \Phi^\mu F x^\nu - F\Phi^\nu x^\mu - \Phi^\nu F x^\mu$ und geben deshalb paarweise Null.

3. *Wechselwirkendes Feld*

Wir beweisen jetzt den zweiten Teil der oben ausgesprochenen Behauptung, daß nämlich die Größe $\Gamma \neq 1$ in (3) oder (4) die Wechselwirkung des Φ-Feldes mit einem anderen Feld in beobachtbarer Weise verändert. Wir nehmen dabei an, daß

* Da bei der Abteilung von (8) bis (10) aus L, (2), keine Vertauschungsregeln verwendet werden, darf man auch weder Φ^μ, Φ, noch $\delta\Phi$ miteinander vertauschen. Beachtet man dies, so entsteht genau die angeschriebene Form von $T^{\mu\nu}$.

das andere Feld ein in konventioneller Weise behandeltes Dirac-Feld sei und daß die Wechselwirkung durch einen ganz konventionellen nichtlinearen Term in der Lagrange-Dichte hervorgerufen sei. Man überzeugt sich dann sofort davon, daß man für die so skizzierte Theorie die alle Wechselwirkungsphänomene beherrschende S-Matrix wie üblich aufschreiben kann; der allgemeine Ausdruck für S hat genau die gleiche Form wie für $\Gamma = 1$.

Erst bei Berechnung spezieller S-Matrix-Elemente tritt ein Unterschied immer dann auf, wenn das betreffende S-Matrix-Element eine oder mehrere Kontraktionen* von Φ-Operatoren enthält. Für eine solche Kontraktion hat man nach den auch hier geltenden Regeln über die Berechnung von S-Matrix-Elementen die Größe

$$<0|P(\Phi(x)\,\Phi(x'))|0> \tag{13}$$

einzusetzen. $P(...)$ bedeutet hier das Dysonsche chronologische Produkt. Durch $|0>$ soll der Vakuumzustand bezeichnet werden, der durch die Gleichung

$$P^{\mu}|0> = 0 \tag{14}$$

und dadurch definiert ist, daß er der Eigenzustand von P_0 mit kleinsten Eigenwerten ist.

Den Vakuum-Erwartungswert (13) kann man wegen (11) ganz genau so berechnen wie in konventionellen Theorien. Wir ersparen uns die Angabe der Rechnung und verweisen stattdessen auf [9] S. 47. Das Resultat ist:

$$<0|P(\Phi(x)\,\Phi(x'))|0> = \tfrac{1}{2}\,D^F(x-x')\,\Gamma_0, \tag{15}$$

mit

$$\Gamma_0 = <0|\Gamma|0> = 1 + <0|F|0>. \tag{16}$$

D^F in (15) ist die konventionelle Fortpfanzungsfunktion für Photonen.

Wegen des Auftretens von D^F in (15) kommen natürlich alle aus der konventionellen Theorie wohlbekannten Singularitäten auch bei uns vor. Der Unterschied zur konventionellen Theorie liegt nur darin, daß bei uns überall $D^F\,\Gamma_0$ statt D^F für Φ-Kontraktionen auftritt. Nun ist es natürlich so, daß durch (7a) die Größe $<0|F|0>$ keineswegs eindeutig gegeben ist. Denn in (7a) kann ja F noch mit beliebigen c-Zahlen multipliziert werden. In Γ_0 steckt also eine völlig willkürliche Zahl.

Auf den ersten Blick könnte man meinen, daß dieses Γ_0 bei der Renormierung eine neue Situation schaffen wird. Jedoch erweist sich diese Vermutung als nicht stichhaltig, weil die Zahl der inneren Φ-Linien eines Graphen der Ordnung n nicht eindeutig von n abhängt; die Renormierungskonstanten treten ja immer in nten-Potenzenauf. Deshalb muß man also shließen, daß unsere neue Konstante Γ_0 eine beobachtbare Abweichung von der konventionellen Theorie hervorbringen sollte,

* In der Sprache der Graphen sind die Kontraktionen „innere Φ-Linien", die virtuellen Φ-Prozessen entsprechen.

4. *Schlussbemerkung*

Verf. ist natürlich keineswegs der Meinung, daß die hier vorgelegte Theorie reale Bedeutung hat. Er ist aber der Meinung, daß die hier gewonnene Erkenntnis prinzipiell wichtig ist: Man kannn solche q-Zahl-Vertauschungsregeln zwischen den Feldgrößen angeben, daß der physikalische Inhalt des isolierten Feldes der übliche bleibt, daß aber die Wechselwirkungen des betrachteten Feldes mit einem anderen Feld in beobachtbarer Weise abgeändert werden. Verf. hofft, daß eine genauere Diskussion dieser durch q-Zahl-Vertauschungsregeln eröffneten Möglichkeiten neue Wege zur Überwindung der Schwierigkeiten quantisierter Feldtheorien freilegen wird.

Anhang

Unser Ansatz (4) für die Vertauschungsregel zwischen den Φ scheint zunächst im Widerspruch zu stehen mit der von Lehmann [10] abgeleiteten allgemeinen Form solcher Regeln. Lehmann hat ja unter sehr allgemeinen Voraussetzungen (Lorentzinvarianz der Theorie, Existenz eines Vakuumzustands und Vollständigkeit des Systems der Eigenzustände des Energie-Impuls-Vierer-Vektors) z. B. gezeigt, daß für skalare Feldgrößen stets

$<0|[\Phi(x), \Phi^{\mu'}(x')]|\,0> = -\,i\delta^{\mu}(x-x') \cdot g$ ($x-x'$ raumartig, $|0>$ = Vakuum-Zustand)

gelten muß, wobei der Zahlenfaktor g von der Feldgleichung abhängt, der man die Größe Φ unterwirft. Für unsere Feldgleichung (1) gilt sogar $g = 1$.

Dieser Widerspruch beruht auf folgendem Umstand: In unserer Theorie wird nicht der gesamte Hilbertraum, auf den die Größen $\Phi(x)$ wirken, von den Eigenvektoren der 4 Operatoren P_{μ} aufgespannt. Denn die Φ enthalten ja, wie aus [4] hervorgeht, noch Bestimmungsstücke der Probekörper, die natürlich ihren eigenen Hilbertraum besitzen. Deshalb erscheint uns der Ansatz

$$\Phi(x) = \Phi(x)\, T(x) \text{ und} \tag{17}$$

$$|0> = |0_{\varphi}> |\, 0_{T}> \tag{18}$$

gerechtfertigt. Hierbei soll Φ nur auf $|0_{\varphi}>$, T nur auf $|0_{T}>$ wirken; $T(x)$ sei der Anteil der Probekörper an $\Phi(x')$. Man kann dann z. B. schreiben:

$$< 0\,|\Phi(x)\,\Phi(x')|\,0> = < 0_{T}|T(x)\,T(x')|\,0_{T}> < 0_{\varphi}\,|\varphi(x)\,\varphi(x')|\,0_{\varphi}>.$$

Nun befriedigt $\varphi(x)$ nach [4] eine kovariante, wenn auch komplizierte Integralgleichung. Nimmt man noch an, daß der mit φ verknüpfte Hilbertraum vollständig durch die Eigenvektoren von $P_{\mu}[\varphi]$ aufgespannt wird, so kann man alle Sätze Lehmanns für den Faktor $<0_{\varphi}\,|\varphi(x)\,\dot{\varphi}(x')|\,0_{\varphi}>$ beweisen und erhält (mit $[T(x),\ T(x')] = 0$) z. B. folgendes Resultat.

$< 0\,|\Phi(x)\,\Phi^{\mu'}(x')|\,0> = -\,i\,\bar{\Gamma}_{0}\,\delta^{\mu}(x-x') \cdot g$, wobei $\bar{\Gamma}_{0} \equiv < 0_{T}|T(x)\ T(x')|\,0_{T}>$ und

g diesmal von der Feldgleichung für die φ abhängt. Es kann natürlich $\bar{\Gamma}_0 = \text{const}$ vorausgesetzt werden, ohne daß dies die Operatorgleichung $T(x) = \text{const.}$ nach sich zieht. $\bar{\Gamma}_0 \cdot g = \Gamma_0$ kann nach unseren Vorstellungen beliebige Zahlenwerte annehmen und ist mit unserem obigen Γ_0 (16) identisch.

Es sei noch ausdrücklich erwähnt, daß man aus den hier skizzierten Uberlegungen keineswegs schließen kann:

$$\Gamma = T(x)\, T(x') \cdot g \quad (\Gamma \text{ aus } (4));$$

es gilt nur die schon angegebene Relation

$$< 0 \mid \Gamma \mid 0 > = g < 0 \mid T(x)\, T(x') \mid 0 >.$$

Γ ist also durch T keineswegs eindeutig bestimmt. Das ist wichtig bei Diskussion der Relationen (7), (7a).

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LABORATORY EQUIPMENT AND TECHNIQUES

AUTOMATIC DEVICE FOR ZONE REFINING OF METALS AND SEMICONDUCTORS

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In this paper the high-efficient automatic arrangement for zone refining of metals and semiconductors is described.

Zone refining was first used for obtaining germanium of a high degree of purity (Pfann 1952).

The method, however, can be used for purifying arbitrary substances, provided they exhibit a difference in concentration of the impurities across the solid/liquid phase boundary (Pfann 1957, Żdanowicz 1957). At present, zone refining is being used especially in the process of preparing very pure metals and semiconductors.

Concerning techniques of zone refining, three modifications should be distinguished (Tanenbaum et al. 1954):

1. Applying a single heating element and transferring the molten zone throughout the entire length of the rod,

2. Applying a number of heating elements, the rod subjected to the process being passed through the array,

3. Applying a number of heating elements in back and forward motion of amplitude d equalling the distance between two consecutive elements.

From simple considerations, it is seen that (3) presents the greatest advantage both with respect to the time required for purifying a rod and to the amount of electric energy involved. Another practical advantage arises from the smaller dimensions of the device.

In preparing semiconductor materials, the authors had to face the problem of applying zone refining both to the elements (e. g. metals with a content of impurities not exceeding $10^{-5}\%$) and to the semiconductors themselves. The necessity arose of designing a simple, economical and easily workable device of relatively high

yield (about 1 kg of the pure substance per week). One wherein resistance heating elements in back and forward motion are applied was found to be the most satisfactory in fulfilling the foregoing requirements.

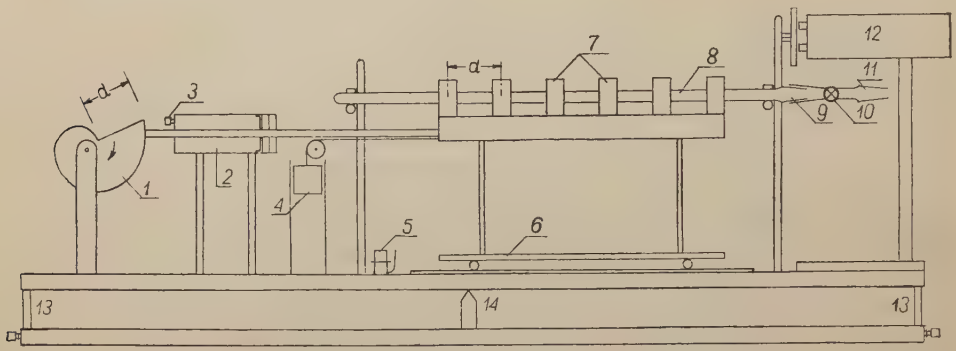

Fig. 1. Diagram of automatic device for zone refining: 1. Spiral leading cam, 2. Air damper, 3. Screw for setting velocity of backward motion, 4. Weight, 5. Connection to counter for recording number of zones, 6. Carriage, 7. Heating elements, 8. Quartz ampoule, 9. Ground quartz/glass connection, 10. Vacuum valve, 11. Ground connection to vacuum system, 12. Electromagnet, 13—14. Suspension, and screws for adjusting position with respect to horizon tal.

Back and forward motion was achieved by applying a spiral leading cam (Fig. 1) rotated at the corresponding rate, and a weight for backward motion. The rate of rotation of the leading cam could be set at from 1/4 to 4 tours per hour. The backward motion could be set at a given velocity with an air damper. The heating elements (two thermally isolated systems of heating elements) were placed on a carriage and connected stiffly to the moving mechanism. According to the thermal properties of the substance

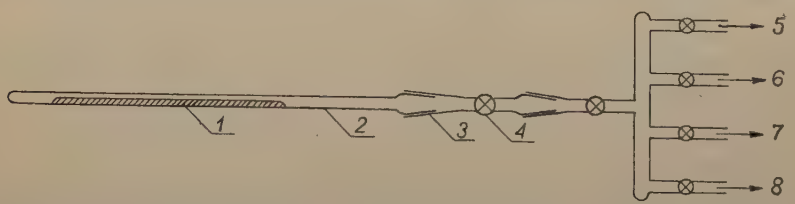

Fig. 2. Diagram of device for preparing ampoules for zone refining: 1. Substance for purification, 2. Quartz ampoule, 3. Ground quartz/glass connection, 4. Vacuum valves, 5. To vacuum system, 6. To hydrogen container, 7. To argon container, 8. To nitrogen container.

to be purified, a system of 8 elements separated by a distance of $d_1 = 6$ cm or one of 6 elements set at a distance of $d_2 = 8$ cm is used. In this way zone refining of rods of about 50 cm in length can be carried out. The substance subjected to purification is placed either within an evacuated quartz ampoule or within one filled with a neutral gas. The quartz ampoule possessed a ground connection and vacuum tap. On evacu-

ating or filling with a neutral gas, the ampoule (Fig. 2) was removed from the vacuum stand; in the process of zone refining, it is not connected with the vacuum pump.

In the process of zone refining the impurities are not distributed homogeneously throughout the molten zone as assumed in theoretical considerations; in fact, a layer with a higher concentration of impurities than that prevailing throughout the remaining part of the molten zone appears within the liquid phase at the liquid/solid phase boundary. This layer affects the value of the coefficient of segregation and reduces considerably (somctimes down to the order of tenths) the effectiveness of the process of zone refining. This is especially true of substances having a low melting temperature, wherein thermal convection is weak. To remove this layer of higher concentration and obtain an homogeneous distribution of the impurities throughout the molten zone, mixing should be applied.

If inductive heating is used, stirring is automatic. With resistance heating, magnetic fields have been used for stirring the molten zone (Pfann, Dorsi 1957). In practice, however, this is no easy procedure.

In the present investigation, intense mechanical stirring was applied. This was achieved by placing the quartz ampoule with the substance to be purified upon elastic steel rods; a source of vibrations was provided by a a.c. fed electromagnet. Thus the molten zone could be transferred at a greater rate without unduly affecting the process of purification.

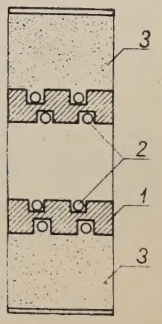

Fig. 3. Cross section through heating element: 1. Ceramic form, 2. Kanthal spirals, 3. Asbestos thermal isolation.

The entire device is placed upon a common support which can be set at various angles with respect to the horizontal. This makes it possible to compensate for the transfer of substance along the sample brought about by the change in density at the melting point, and to obtain rods of constant cross section as a result of zone refining.

Fig. 2. shows a diagram of the used for preparing ampoules for the process of zone refining. If required, the process can be carried out in vacuo or under pressure in an atmosphere of neutral gas. The quartz ampoules had a diameter of 20 mm.

Fig. 3 brings a cross section through the heating elements. These were made of spirals of kanthal wire 1 mm in diameter. The maximum power output per single heating element was 320 W.

The authors' device as tested over one thousand working hours proved to be entirely reliable and to fulfill the requirements already discussed. Once in every 24 hours, an ampoule containing a new portion of the substance has to be placed in the device.

Zone refining of some metals (Sb, Cd, In, Te, Zn, Pb, Bi) and semiconductors (Ge, InSb, GaSb, HgTe, HgSe, In_2Te_3) has been carried out satisfactorily.

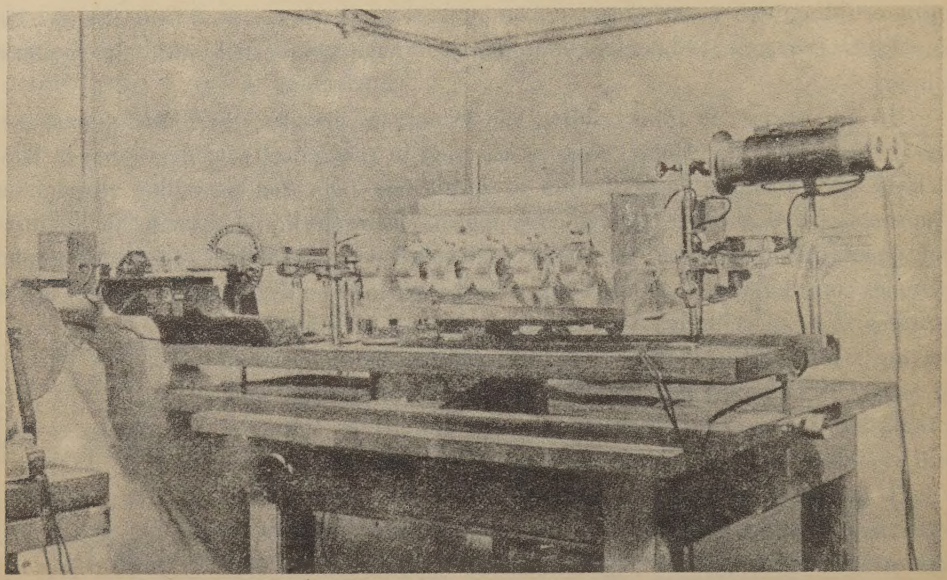

Fig. 4. Picture of automatic device for zone refining.

When purifying substances having a low point of fusion and high thermal conductivity (e. g., In), water cooling is applied.

The present investigation was carried out under the direction of Professor Dr L. Sosnowski.

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